Reminder

Groth. Spectral sequence:

$$\mathcal{C} \xrightarrow{F} \mathcal{C}' \xrightarrow{G} \mathcal{C}''$$

additive, left exact between Abelian cats w/ $\mathcal{C}, \mathcal{C}'$ having enough injectives, and suppose that F takes injectives to G-acyclics.

(i.e. if $I \in \mathrm{ob}(\mathcal{C})$ injective then $R^q G(F(I)) = 0 \quad q > 0$)

then we get a (cohomological) spectral sequence

$$E_2^{pq} = (R^p G)(R^q F(A)) \Rightarrow R^{p+q}(G \circ F)(A)$$

$$\| $$

$$E^{p+q}(A)$$

Reminder: $\mathcal{C}$ a site, $\mathrm{AbPre}(\mathcal{C})$ ab. presheaves

$\mathrm{AbShv}(\mathcal{C})$ ab sheaves.

$$\mathrm{AbShv}(\mathcal{C}) \xrightarrow{\Gamma} \mathrm{Ab}$$

$$\mathcal{F} \longmapsto \Gamma(\mathcal{F})$$

Def $H^i(\mathcal{C}, \mathcal{F}) = R^i\Gamma(\mathcal{F})$

Def $H^i(U, \mathcal{F}) = R^i\Gamma(U, \mathcal{F})$

$$\mathrm{AbShv}(\mathcal{C}) \xrightarrow{\iota} \mathrm{AbPre}(\mathcal{C})$$

$$\mathrm{AbPre}(\mathcal{C}) \xrightarrow{\check{H}^0(\{U_i \to U\})} \mathrm{Ab}$$

both left exact.

$\{U_i \to U\} \in \mathrm{Cov}(\mathcal{C})$

$$\begin{tikzcd} \mathrm{AbShv}(\mathcal{C}) \arrow[r] \arrow[rr, bend left, "{\Gamma(U, \mathcal{F})}"] & \mathrm{AbPre}(\mathcal{C}) \arrow[r] & \mathrm{Ab} \\ \mathcal{F} \arrow[r, mapsto] & \mathcal{F} \arrow[r] & \check{H}^0(\{U_i \to U\}, \mathcal{F}) \end{tikzcd}$$

$$\mathcal{F}(U) = \ker\Big(\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_{ij})\Big)$$

Claim: if $\mathcal{I} \in \mathrm{AbShv}(\mathcal{C})$ is injective $\Rightarrow$ assoc. presheaf is Acyclic.

Sheafification:

$$\mathrm{Pre}(\mathcal{C}) \xrightarrow{\#} \mathrm{Shv}(\mathcal{C})$$

$$\mathrm{Pre}(\mathcal{C}) \xrightarrow{+} \mathrm{Pre}(\mathcal{C})$$

$\# = +^2$ $\quad ++ = \#$

Def $\mathcal{F}^+(U) = \varinjlim_{\{U_i \to U\}} \check{H}^0(\{U_i \to U\}, \mathcal{F})$

$\check{H}^0$ = local sections, which agree.

Prop: $+(\mathcal{F})$ is always a separated presheaf

($\mathcal{G}$ sep $\Leftrightarrow$ $\mathcal{G}(U) \to \check{H}^0(\{U_i \to U\}, \mathcal{G})$ is mono)

i, $++$ is always a sheaf.

ii, if $\mathcal{F} \to \mathcal{G}$, $\mathcal{G}$ a sheaf

$\exists!$ $\mathcal{F}^+ \to \mathcal{G}$ s.t.

$$\begin{tikzcd} \mathcal{F} \arrow[rr] \arrow[dr] & & \mathcal{G} \\ & \mathcal{F}^{++} \arrow[ur] & \end{tikzcd}$$
commutes.

Prop: # is exact. and is left adj to i

Gen nonsense $\Rightarrow$ if a functor has an exact left adjoint then it preserves injectives.

$\Rightarrow$ get Čech spectral seq:

$$\left(R^p \check{H}^0(\{U_i \to U\})\right) \circ \left(R^q i \mathcal{F}\right)$$
$$\Rightarrow R^{p+q} \Gamma(U, -)(\mathcal{F})$$

$$\check{H}^p(\{U_i \to U\}, \underbrace{R^q i\mathcal{F}}) \Rightarrow H^{p+q}(U, \mathcal{F})$$

Claim: $R^q i = \mathcal{H}^q$

where $\mathcal{H}^q(\mathcal{F})(U) = H^q(U, \mathcal{F})$

via univ δ-functors

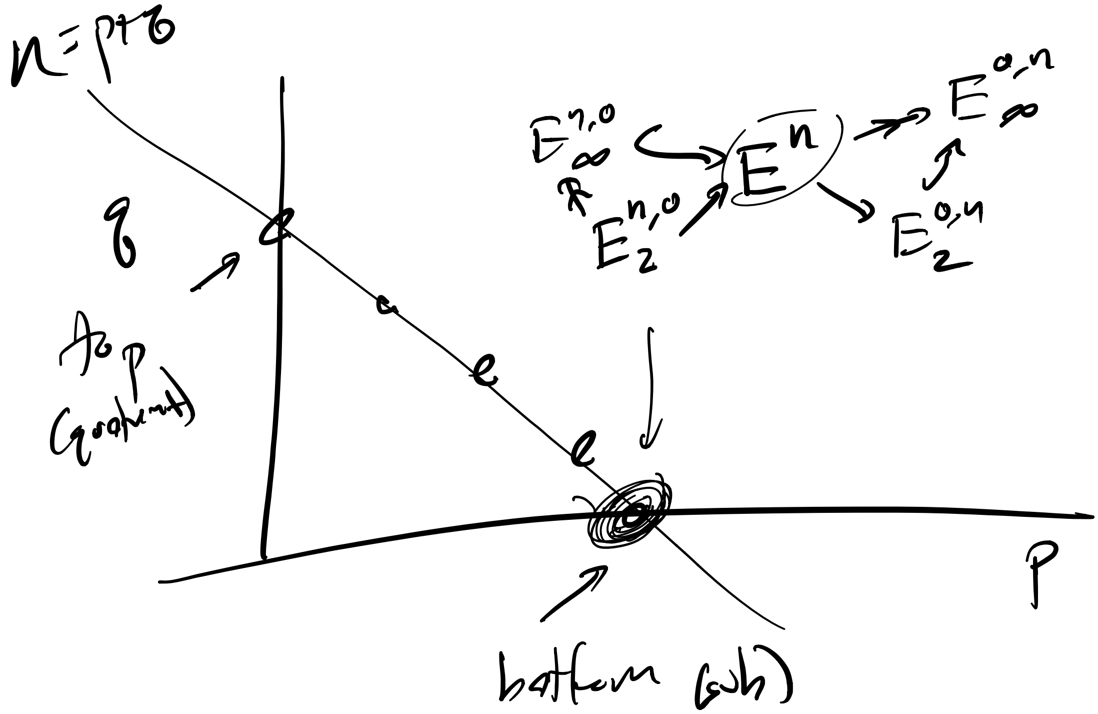
Čech SS:

$$\check{H}^p(\{U_i \to U\}, \mathcal{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(U, \mathcal{F})$$

$$\check{H}^n(\{U_i \to U\}, \mathcal{H}^0(\mathcal{F})) \to H^n(U, \mathcal{F})$$

$$H^n(U, \mathcal{F}) \to \check{H}^0(\{U_i \to U\}, \mathcal{H}^n(\mathcal{F}))$$

$$\ker\left(\prod H^n(U_i) \rightrightarrows \prod_{ij} H^n(U_{ij})\right)$$



$n=p+q$
q
top p
(quotient)
p
bottom (sub)
$E^{n,0}_\infty \hookrightarrow E^n \twoheadrightarrow E^{0,n}_\infty$
$E^{n,0}_2$
$E^{0,n}_2$