Op(X)   op(Y)

If $\mathcal{C}$ & $\mathcal{D}$ are Groth. tops

morphism of sites is a functor $F: \mathcal{C} \to \mathcal{D}$

s.t. - - - -

a morphism of sites is a functor $f^{-1}: \mathcal{C} \to \mathcal{D}$ s.t. ...

$f: \mathcal{D} \to \mathcal{C}$ $f: Y \to X$

Last time: $\{U_i \to U\}$ cover in $\mathcal{T}$

defined $\check{H}^i(\{U_i \to U\}, \mathcal{F})$ $\mathcal{F}$ presheaf.

$\simeq$ r. derived functors at $\check{H}^0(\ )$

$\underline{\mathrm{AbPre}(\mathcal{T})} \to \underline{\mathrm{Ab}}$

Can also take limits of refined covers,

get $\check{H}^i(U, \mathcal{F}) = \lim\limits_{\{U_i \to U\}} \check{H}^i(\{U_i \to U\}, \mathcal{F})$

also r. derived functors as above.

Čech spectral seq.

$$\check{H}^p(\{U_i \to U\}, \mathcal{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(U, \mathcal{F})$$

presheaf $\mathcal{H}^q(\mathcal{F})(V)$

$= H^q(V, \mathcal{F})$

similarly, $\check{H}^p(\mathcal{U}, \mathcal{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(U, \mathcal{F})$

τ = Zariski topology, if $\{U_i \to U\}$ affine cover, U separated

$U_{i_1} \cap \dots \cap U_{i_k}$ affine.

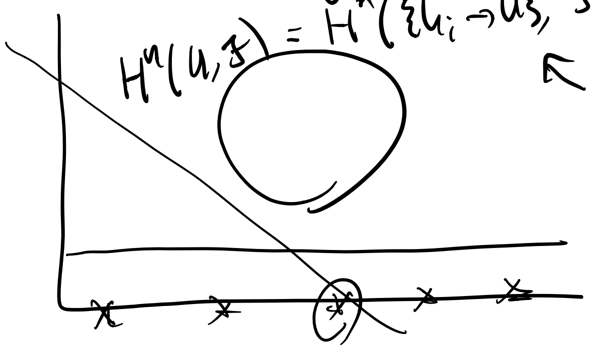
$\mathcal{F}$ = q.coherent $\quad H^q(V, \mathcal{F}) = 0 \quad V$ affine

$$\mathcal{H}^q(\mathcal{F})(U_{i_1} \cap \dots \cap U_{i_k}) = 0$$

$$\check{H}^p(\{U_i \to U\}, \mathcal{H}^q(\mathcal{F})) = 0 \text{ if } q > 0$$

$$H^n(U, \mathcal{F}) = \check{H}^n(\{U_i \to U\}, \mathcal{F})$$

Standard result in Hartshorne to compute q.coh cohomology.



Remark: (Artin)

<u>Theorem</u> X q.compact s.t. any finite subset is contained in an affine (e.g. X q.projective) and $\mathcal{F}$ a sheaf in X_{et} then $\check{H}_{et}(X, \mathcal{F}) = H_{et}(X, \mathcal{F})$

Exercise if cover is $\{U_i \to U\}$ $i = 1,2$

Zariski topology

then Čech spectral sequence ⇝ Mayer–Vietoris sequence

$$\begin{tikzcd} \to H^n(U,\mathcal{F}) \arrow[r] & H^n(U_1,\mathcal{F}) \oplus H^n(U_2,\mathcal{F}) \arrow[r] & H^n(U_1 \cap U_2, \mathcal{F}) \arrow[dl] \\ & H^{n+1}(U,\mathcal{F}) & \end{tikzcd}$$

Leray Sequence

let $f : T \to T'$ be a morphism of sites, $U \in T$

and $\mathcal{F}'$ a sheaf on T' then have a spec. seq.

$$E_2^{p,q} = H^p(U, R^q f^s(\mathcal{F}')) \Rightarrow H^{p+q}(f(U), \mathcal{F}')$$

$f^p(\mathcal{F}')(U) = \mathcal{F}'(f(U))$ (pullback to presheaves)

$$\begin{tikzcd} \mathrm{PreShv} \arrow[r, bend left, "\#"] & \mathrm{Shv} \arrow[l, bend left, "i"] \end{tikzcd}$$

$$f^s = \# \circ f^p \circ i$$
$$= f^p \circ i$$

in cases of interest, $\#$ will not be necessary.

$$E_2^{p,q} = H^p(U, R^q f^s(\mathcal{F}')) \Rightarrow H^{p+q}(f(U), \mathcal{F}')$$

also: $R^q f^s(\mathcal{F}') \simeq \left(f^p \mathcal{H}^q(\mathcal{F}')\right)^\#$

$\mathcal{F}/$

$$\begin{array}{ccc} X & & \mathcal{O}_p(X) \\ \downarrow g & & \uparrow g^{-1} = f \\ \mathcal{O}_V \subseteq Y & & \mathcal{O}_p(Y) \end{array}$$

$$H^p(Y, \underbrace{R^q f^s(\mathcal{F})}_{\left(f^p \mathcal{H}^q(\mathcal{F})\right)^\#}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

$$\begin{aligned} f^p \mathcal{H}^q(\mathcal{F})(V) &= \mathcal{H}^q(\mathcal{F})(g^{-1}V) \\ &= H^q(g^{-1}V, \mathcal{F}) \end{aligned}$$

cohom of space X $\longleftrightarrow$ cohom of base Y w/ coeffs in cohom of fibers

Famous example: Hocshild - Serre spectral sequence in group cohomology

$$\tau' \longleftarrow \tau \qquad G \qquad G/N$$

G-sets G/N - sets N

Descent :

$\{U_i \to U\}$ cover then

maps between sheaves on U

$\Longleftrightarrow$

maps between pull backs to U_i's which agree on overlaps

$$\begin{array}{c} \mathrm{Hom}_U(\mathcal{F}, \mathcal{G}) \\ \downarrow \\ \mathrm{Hom}_{U_i}(\mathcal{F}|_{U_i}, \mathcal{G}|_{U_i}) \\ \downarrow\downarrow \\ \mathrm{Hom}_{U_i}(\mathcal{F}|_{U_{ij}}, \mathcal{G}|_{U_{ij}}) \end{array}$$

if U scheme w/ Grothendieck top (étale or other)

want this to hold for q. coherent sheaves.

note: if $\mathcal{F}$ a q. coh sheaf on U

$U_i \to U$, have pull back of q. coherent sheaves

$\rightsquigarrow$ gives a presheaf on cat of $\mathrm{Schemes}/U$

can ask if this is a sheaf w/r/t to some top on