Facts (Etale cohomology of a field)

F field $F_{ét}$ = étale site of F
small

Eq of cats

$G = Gal(F^{sep}/F)$

$$\begin{tikzcd} Shv(F_{ét}) \arrow[r, leftrightarrow] & \text{Discrete } G\text{-Sets} \end{tikzcd}$$

$\mathcal{F} \longmapsto \mathcal{F}(L) \circlearrowleft_{Gal(L/F)}$

$ob(F_{ét}) = (Spec L \to Spec F)$

$L = \prod_{i=1}^{n} L_i$

L_i/F sep. ext.

covs: nonempty fam. L

$\searrow$? eq.

$Shv(F_{ét}^{v.sm}) \longleftarrow\!\shortmid M$

$\mathcal{F}_M(L) = M^H$

$$\begin{tikzcd} F^{sep} \arrow[d, dash, "H"] \\ L \arrow[d, dash] \\ F \end{tikzcd}$$

G

$\mathcal{F}_M(\prod L_i) = \prod M^{H_i}$

L_i as

def

$F_{ét}^{v.small}$ "

as in $F_{ét}$ but w/ only L/F as here

$F_{ét}^{v.small} \longrightarrow F_{ét}$

$\uparrow$ eq. of cats.

$$\begin{tikzcd} Shv(F_{ét}) \arrow[r] & \text{Disc. } G\text{-Sets} \end{tikzcd}$$

$\mathcal{F} \longmapsto$ "$\mathcal{F}(F^{sep})$"

$\varinjlim_{F \subset L \subset F^{sep}} \mathcal{F}(L)$

$$\begin{tikzcd}
\mathcal{F} \arrow[rrr, mapsto, bend left] & \mathrm{Shv}(F_{\acute{e}t}) \arrow[r, shift left] \arrow[d, "\Gamma"] & \text{Disc. } G\text{-set} \arrow[l, shift left] \arrow[d, "(\ )^G"] & M \arrow[d, mapsto] \\
\Gamma(\mathcal{F}) \arrow[d, phantom, "\shortparallel"] & \text{Sets} \arrow[r, equal] & \text{Set} & M^G \\
\mathcal{F}(F) \arrow[rrru, mapsto, bend right] & & &
\end{tikzcd}$$

$\mathrm{Ab}\,\mathrm{Shv}(F_{\acute{e}t}) \longleftrightarrow \text{Disc } G\text{-Mods}$

$$R^i\Gamma = R^i(\ )^G$$

$$H^i(F_{\acute{e}t}, \mathcal{F}) \xleftarrow{\sim} H^i(\mathrm{Gal}(F^{sep}/F), \mathcal{F}(F^{sep}))$$

choosing sep closure gives a canonical iso

$\mathcal{F} = \mu_n$ $\quad (n, \overset{\text{all}}{\text{chars}}\ X)$ $\quad X$ scheme.

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{\cdot n} \mathbb{G}_m \to 1$$

sheaves on big or small étale site on X

$$\mu_n(Y) = \{ f \in \mathcal{O}_Y(Y)^* \mid f^n = 1 \}$$

$$\mathbb{G}_m(Y) = \mathcal{O}_Y(Y)^*$$

$\mathbb{G}_m = \mathbb{G}_{m/X} = \operatorname{Spec} \mathbb{A}^1_X \setminus X$

$\nearrow$ at 0.

locally $X = \operatorname{Spec} R$ $\operatorname{Spec} R[t, t^{-1}]$

$\mu_n \hookrightarrow \mathbb{G}_m$ $\mu_{n,R} = \operatorname{Spec} R[t]/t^n - 1$

Points in étale site:

if X is a scheme, $x \in X$ scheme theoretic pt

consider 'stalks at x'

limits of "covers at x."

limits of étale morphisms w/ x in image.

exactness for sheaves on X only needs to be checked for sections in each of these limits for each x.

$$\begin{tikzcd}
 & U \arrow[r] & \underset{x}{X} \\
W \arrow[ur] \arrow[dr] & & \\
 & V \arrow[uur] &
\end{tikzcd}$$

can find a "limiting" cofinal $\nearrow$ family of covers

$$\cdots \to U_2 \to U_1 \to X$$

with all U_i's connected immed.

in limit, get a localiz.

Def: A strictly local rg is a Henselian local rg w/ separably closed residue field.

Hensel's lemma holds

How to think of this:

take Zariski local rg $R = \mathcal{O}_{X,x}$

Force Hensel to hold:

for each poly $f \in R[x]$ w/ $\bar{f} \in R/m[x]$ sep., α a root of $\bar{f}$

adjoin a root a of f restricting to α.

get a Henselian rg R^h

res ext k^{sep}/k ⇝ rg ext R^{sh}/R^h unram.

R^{sh} = strictly local rg at x.

way back

Kummer seq

$$\begin{tikzcd}
 & \mu_n(X) & \mathbb{G}_m(X) & \mathbb{G}_m(X) \\
0 \arrow[r] & H^0(X,\mu_n) \arrow[r] & H^0(X,\mathbb{G}_m) \arrow[r] & H^0(X,\mathbb{G}_m) \arrow[dll] \\
 & H^1(X,\mu_n) \arrow[r] & H^1(X,\mathbb{G}_m) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H^1(X,\mathbb{G}_m) \arrow[d, phantom, "\shortparallel"] \\
 & & \operatorname{Pic} X & \operatorname{Pic} X
\end{tikzcd}$$

$$0 \to \mathbb{G}_m(X)/n \longrightarrow H^1(X, \mu_n) \to \operatorname{Pic} X[n] \to 0$$

$$X = \operatorname{Spec} F$$

$$F^*/(F^*)^n \simeq H^1(F, \mu_n)$$