Idea of a torsor:

"relative principal homogeneous space"

principal hom. space:

if G a group X a PHS for G means

G acts simply transitively on X.

arithmetic example:

$G = \mathbb{Z}/2\mathbb{Z}$ $\qquad X =$ "roots of 2 in $\mathbb{Q}$"

$$X = \operatorname{Spec} \frac{\mathbb{Q}[x]}{x^2-2} \circlearrowleft x \mapsto -x$$

$X(\mathbb{Q})$ $\qquad L = \mathbb{Q}(\sqrt{2})$

looking "locally at X" $\qquad X_L \longrightarrow X$ étale cover

$$e_1 \longleftrightarrow e_2$$

$$L[x]/x^2-2 \to \underset{L}{\underbrace{L[x]/x-\sqrt{2}}} \times \underset{L}{\underbrace{L[x]/x+\sqrt{2}}} \qquad \operatorname{Spec} L[x]/x^2-2$$

$$\cong$$

$x \longmapsto \sqrt{2}, -\sqrt{2}$ $\qquad \operatorname{Spec} L \sqcup \operatorname{Spec} L$

$-x \longmapsto -\sqrt{2}, \sqrt{2}$ $\qquad \mathbb{Z}/2\mathbb{Z}$

$G \longmapsto$ group scheme$/\mathbb{Q}$ $\qquad G \times G \to G$

$\mathbb{Z}/2 \rightsquigarrow \underset{e}{\operatorname{Spec}\mathbb{Q}} \sqcup \underset{\sigma}{\operatorname{Spec}\mathbb{Q}}$

$G_L \simeq X_L$

Top example

$$\begin{tikzcd} G \circlearrowright \tilde{X} \arrow[d, "\pi"] \\ X \end{tikzcd}$$

top cov space regular Galois

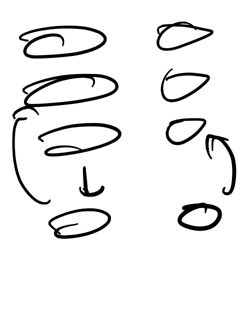
G = gp of deck transformations.

$\pi \rightsquigarrow \underset{\text{on } X}{\mathcal{F}}$ $\qquad \mathcal{F}(U) = \{ s : U \underset{\text{cont.}}{\to} \tilde{X} \text{ sections of } \pi \}$

$G(U) = \{ f : U \to G \text{ cont.} \}$

$G(U) \circlearrowright \mathcal{F}(U)$ $\quad$ cov space $\Rightarrow$ U suff small

this action is simply transitive



ex:

Analysis

$$\begin{array}{c} \mathbb{C}\setminus\{0\} \\ \Big\downarrow z \mapsto z^2 \\ \mathbb{C}\setminus\{0\} \end{array}$$

sections on open U are square root fns.

$$\begin{array}{l} \mathbb{C}(\sqrt{x}) \\ \quad \vert \\ \mathbb{C}(x) \end{array}$$ meromorphic fns on $\hat{\mathbb{C}}$

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sheaf notion

$\mathcal{G}$: sheaf of groups on a site $\mathcal{T}$

functor $\mathcal{G}: \mathcal{T} \to \mathrm{Group}$ s.t. forget $\rightsquigarrow$ sets sheaf.

$\mathcal{X}$

$h_G'' \leftarrow G$

and $\mathcal{X}$ a sheaf of sets

$\mathcal{X}$ is a $\mathcal{G}$ torsor if we are given an action of $\mathcal{G}$ on $\mathcal{X}$

i.e. $\mathcal{G} \times \mathcal{X} \xrightarrow{a} \mathcal{X}$

s.t. for any $U \in \mathcal{T}$ $\exists$ cover $\{U_i \to U\}$ s.t. $\mathcal{X}_{U_i} \simeq \mathcal{G}_{U_i}$ as sheaves of sets w/ $\mathcal{G}_{U_i}$ action.

scheme notion

top on some "big" S-site.

G: a group scheme $/S$

$G \times G \to G$.

a G-torsor is a S-scheme X w/ G action on it

$G \times X \xrightarrow{a} X$

$X = X_S$

s.t. locally on S, looks like G acting on itself.

$\exists \{S_i \to S\}$ cover in $\mathcal{T}$ s.t. $G_{S_i} \simeq X_{S_i}$ as schemes w/ G_{S_i} action.

if $\mathcal{X} \simeq \mathcal{G}$ we say $\mathcal{X}$ is formal.

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action $\mathcal{G} \times \mathcal{X} \xrightarrow{a} \mathcal{X}$ all U

$$\mathcal{G}(U) \times \mathcal{X}(U) \xrightarrow{a(U)} \mathcal{X}(U)$$

s.t.

$$\begin{tikzcd} \mathcal{G} \times \mathcal{G} \times \mathcal{X} \arrow[r, "m"] \arrow[d, "\mathrm{id} \times a"'] & \mathcal{G} \times X \arrow[d, "a"] \\ \mathcal{G} \times X \arrow[r, "a"'] & X \end{tikzcd}$$

$e \in \mathcal{G}(U)$ acts formally

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i.e. $G_{S_i} \xrightarrow{\varphi} X_{S_i}$

s.t.

$$\begin{tikzcd} G_{S_i} \times G_{S_i} \arrow[r, "m"] \arrow[d, "\mathrm{id} \times \varphi"'] \arrow[dr, phantom, "\circlearrowright"] & G_{S_i} \arrow[d, "\varphi"] \\ G_{S_i} \times X_{S_i} \arrow[r, "a"'] & X_{S_i} \end{tikzcd}$$

Def 3 of torsor /S base $\quad G \times X \xrightarrow{a} X$

G gp scheme X scheme w/ G action

we say X is a G torsor if

$$G \times X \xrightarrow{a \times \pi_2} X \times X \text{ is an isomorphism.}$$

ex: X is a def 3 G-torsor iff it is a G torsor in any top τ for which $X \to S$ is a cover.

$G_X \quad X_X$

if $X \to S$ is a cov and X is a G torsor

$\Downarrow$

$G \times X \xrightarrow{\sim} X \times X$

$\swarrow$

$X_X \simeq G_X$