Convention:

Groups should act on the right

(right G-torsors)

$$X \times G \xrightarrow{a} X$$

Given a G torsor X (all $/S$) w/r/t to some top τ

$\Rightarrow \{S_i \to S\}$ cov, isoms

$$\varphi_i : G_{S_i} \xrightarrow{\sim} X_{S_i}$$

via descent, S-schemes

$\updownarrow$

Descent data for schemes w/r/to cov $\{S_i \to S\}$

i.e. $$Y/S \longleftrightarrow \Big(Y_i/S_i,\ \psi_{ij} : Y_i|_{\underbrace{S_i \times_S S_j}_{S_{ij}}} \xrightarrow{\sim} Y_j|_{S_{ij}} \Big)$$

s.t. $\psi_{ik} = \psi_{jk} \psi_{ij}$

$$X/S \longleftrightarrow \Big(X_{S_i}/S_i,\ X_{S_i}|_{S_{ij}} \xrightarrow{\sim} X_{S_j}|_{S_{ij}} \Big)$$

$$\varphi_i \uparrow \wr \qquad \wr \qquad X_{S_i}|_{S_{ij}} = X_{S_{ij}} \xrightarrow{id} X_{S_{ij}} = X_{S_j}|_{S_{ij}}$$

$$\varphi_i|_{ij} \qquad \varphi_j|_{ij}$$

$$R \otimes_A B \otimes_B C = R \otimes_A C$$

$$\Big(G_{S_i}/S_i,\ G_{S_i}|_{S_{ij}} \xrightarrow{\varphi_j|_{ij}^{-1} \circ \varphi_i|_{ij}} G_{S_j}|_{S_{ij}} \Big)$$

$$G_{S_i}|_{S_{ij}} = G_{S_{ij}} \qquad G_{S_j}|_{S_{ij}} = G_{S_{ij}}$$

not an orb. scheme map

φ_i φ_j are isom. of sheaves w/ G action.

so this is a morphism commutes w/ R right G action.

$$\mathrm{Hom}_{r\text{-}G_{S_{ij}}\text{-schemes}}(G_{S_{ij}}, G_{S_{ij}})$$

$$\mathrm{Hom}_{r\text{-}G\,sch}(G, G) = G(T)$$

G/T have

$$\begin{tikzcd} & G \arrow[r, "\alpha"] & G \\ T' \arrow[ur] \arrow[dr] & & \\ & T \arrow[uu, "e"] \arrow[uur, "g_0"'] & \end{tikzcd}$$

$$\begin{tikzcd} G\times G \arrow[r, "a"] \arrow[d, "\alpha\times id"'] & G \arrow[d, "\alpha"] \\ G\times G \arrow[r, "m"'] & G \end{tikzcd}$$

$$G \xrightarrow{\alpha} G$$

$$G\times G \xrightarrow{a} G$$

$\alpha(gh) = \alpha(g)h$

$\alpha(e) = g_0$ $\quad\alpha(g) = \alpha(eg) = \alpha(e)g$
$= g_0 g$

$$G(T')\times G(T') \xrightarrow{a} G(T')$$

$\alpha_{T'}$ = left mult by g_0 = $a(e)$

$\alpha: G \to G$ is left mult by g_0

$$X/S \longleftrightarrow (X_{S_i}, id)$$

$$\wr$$

$$(G_{S_i}, G_{S_{ij}} \xrightarrow{g_{ij}} G_{S_{ij}})$$

$G(S_{ij})$ left mult by

$$g_{ij} = \varphi_j|_{ij}^{-1}\, \varphi_i|_{ij} \qquad \underbrace{g_{ik}|_{S_{ijk}} = g_{jk}|_{ijk}\, g_{ij}|_{ijk}}$$

<u>Define</u> $\check{Z}^1(\{S_i \to S\}, G)$ $\qquad g_{ij} - g_{ik} + g_{jk} = 0$

$$= \{(g_{ij})_{ij} \mid g_{ij} \in G(S_{ij}),\ g_{ik}|_{ijk} = g_{jk}|_{ijk}\, g_{ij}|_{ijk}\}$$

Could change φ_i by $\varphi_i g_i^{-1}$

changes to equiv g_{ij}

$$g_{ij} = \varphi_j|_{ij}^{-1}\, \varphi_i|_{ij} \qquad g'_{ij} = (\varphi_j g_j^{-1})|_{ij}^{-1}\, (\varphi_i g_i^{-1})|_{ij}$$

$$= g_j \varphi_j^{-1} \varphi_i g_i^{-1}$$

$$= g_j g_{ij} g_i^{-1}$$

we define eq. rel on $\check{Z}(\{S_i \to S\}, G)$

by $(g_{ij}) \sim (g'_{ij})$ iff $\exists (g_i)$, $g_i \in G(S_i)$

s.t. $g'_{ij} = g_j g_{ij} g_i^{-1}$

<u>Def</u> $\check{H}^1(\{S_i \to S\}, G) = \check{Z}^1(\{S_i \to S\}, G)/\sim$

and $\check{H}^1(S, G)$ $\check{Z}^1(S, G)$

as limit over all covers.

We've "shown"

have a bijection between

$\left\{ G\text{-torsors w/ a trivialization on } \{S_i \to S\} \right\}$ iso. classes of

$\updownarrow$

$\check{H}^1(\{S_i \to S\}, G)$

and so $\{G\text{-torsors } / S \text{ w/r/to } \tau\}$

$\updownarrow$

$\check{H}^1(S_\tau, G)$

<u>Examples</u> $G = \mathbb{G}_m$ (τ = fpqc, fppf, étale, Zariski)

In this case, $\mathbb{G}_m = \mathrm{Aut}(\mathcal{O}_X)$ as a coherent sheaf

$$\overset{\shortparallel}{\mathcal{O}_X^*} \qquad R \xrightarrow{\sim} R$$

as an R-mod

$\longleftrightarrow R^*$

descent data for a $\mathbb{G}_m$-torsor

" descent data for a coherent sheaf which is loc. isom. to $\mathcal{O}_X$

(i.e. rk 1 locally free sheaf)

$\check{H}(S, \mathbb{G}_m) = \{$ iso. classes of loc. free shvs of rk 1, trivializable in τ locally. $\}$

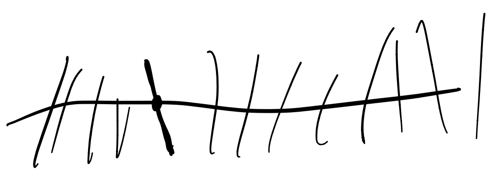
Ring thy: locally free = projective module

proj / locally = free.

projective after faithfully flat base chg = projective.

$$\begin{tikzcd}
\mathbb{G}_m\text{-torsor} \arrow[r] \arrow[d, leftrightarrow, "\wr\ \text{eq}"'] & (\mathbb{G}_{m,S_i}, \varphi_{ij}) \\
\text{Line bundles} \arrow[r, leftrightarrow] & (\mathcal{O}_{S_i}, \varphi_{ij}) \text{ coh-shf glg data}
\end{tikzcd}$$

$\varphi_{ij} = \mathbb{G}_m(S_{ij}) = \mathrm{Aut}_{S_{ij}}(\mathcal{O}_{S_{ij}})$



S

$\mathbb{A}^1 \setminus \{0\} = \mathbb{G}_m$