Brauer group

F field

<u>Def</u> An F-central simple algebra is an associative unital, not necessarily commutative F-algebra such that

- $\dim_F A < \infty$, $Z(A) = F$
- A has no nontrivial 2-sided ideals.

<u>Examples</u> $A = F$, $A = M_n(F)$, $A = \mathbb{H}/\mathbb{R}$

<u>Fact</u> Wedderburn-Artin: $CSA/F \implies A \simeq M_m(D)$
D a CSA which is division. (CDA)

A is a $CSA/F \implies A \otimes_F F^{sep} \simeq M_n(F^{sep})$
(i.e. A is a twisted form of the matrix algebra $M_n(F)$ some n)

$\Downarrow$

iso classes in bijection w/
$H^1(F, \mathrm{Aut}(M_n(F^{sep})))$
$= H^1(F, PGL_n)$
$= H^1(\mathrm{Gal}(F^{sep}/F), PGL_n(F^{sep}))$

$\mathrm{Aut}\, M_n(F) \twoheadleftarrow GL_n(F) \leftarrow F^\times$
$(S \mapsto TST^{-1}) \longleftarrow T$

Can check: $B \otimes_F M_n(F) = M_n(B)$
$M_m(F) \otimes M_n(F) = M_m(M_n(F)) = M_{nm}(F)$

$\Rightarrow$ CSA $\otimes$ CSA = CSA. (by descent)

Def If A, B are CSA/F we say $A \sim B$ (Brauer equivalence)

if same "underlying division algebra" i.e.

$A \simeq M_s(D)$ $\quad$ $B \simeq M_t(D')$ $\quad$ D, D' CDA/F

& $D \simeq D'$ $\quad$ (in W-A, D is unique up to iso)

$A \sim D$ if D is $\simeq$ to underlying d.a. of A

Define $\quad [A] + [B] = [A \otimes B]$.

gives a group structure $\quad Br(F)$

Inverses: recall A^{op} is the alg. w/ same elements

and multiplication $\circ^{op}$ via $a \circ^{op} b = ba$

important fact: left A^{op} modules $\longleftrightarrow$ right A-modules

If A is a CSA then

$$A \otimes A^{op} \longrightarrow End_F(A)$$

$$a \otimes b \longmapsto (x \mapsto axb)$$

is an iso since $1 \longrightarrow 1$, dim same.

and ker is an ideal. ✓ $\quad [A^{op}] = -[A]$

Crossed products

Given a finite Galois extension $\begin{array}{c} E \\ \big| G \\ F \end{array}$ (E can be a commutative ring (not a field))

and $c : G \times G \longrightarrow E^*$, can form the algebra

$$(E, G, c) = \bigoplus_{\sigma \in G} E u_\sigma$$

$$u_\sigma u_\tau = c(\sigma, \tau) u_{\sigma\tau} \qquad u_\sigma x = \sigma(x) u_\sigma \quad x \in E$$

(non unital, non associative...)

but c satisfies 2-cocycle condition $\Leftrightarrow$ associative (and then unital)

in this case, we call (E, G, c) a crossed product algebra.

FACT: (E, G, c) is a CSA if a crossed product.

Fun fact: E/F is G-Galois $\Leftrightarrow$ $\exists c$ s.t. get a CSA

$\Leftrightarrow$ is a CSA for $c = 1$ (Dedekind lemma)

<u>FACT:</u> Every CSA/F is Brauer equivalent to a crossed product algebra!

<u>Deep FACT:</u> All division algebras (CDA) over local or global fields are crossed products for cyclic Gal extensions "cyclic algebras"

If $G = \langle \sigma \mid \sigma^n \rangle$ E/F G-Galois, can find a basis for any crossed product $(E, G, c) = \bigoplus E u_\sigma$ consisting of x_σ's s.t. $(x_\sigma = \lambda_\sigma u_\sigma)$ $\lambda_\sigma \in E^*$

$$(E, G, c) = \bigoplus E x_\sigma$$

$$x_{\sigma^i} x_{\sigma^j} = \begin{cases} x_{\sigma^{i+j}} & \text{if } i+j<n \\ b\, x_{\sigma^{i+j-n}} & \text{if } i+j \geq n \end{cases}$$

same $b \in F^*$

Denote this presentation by (E, σ, b) "cyclic algebra"

Moreover:

$$(E, \sigma, b) \otimes (E, \sigma, b') \sim (E, \sigma, bb')$$

$$\{E\text{-cyclic alg's}\}/\sim \; < \; Br(F)$$

Notation: $\mathrm{Br}'(E/F)$ $\qquad F^* \twoheadrightarrow \mathrm{Br}(E/F)$

<u>Thm</u> Have an iso: $\mathrm{Br}(E/F) \simeq F^*/N_{E/F}(E^*)$

<u>ex:</u> if $E = F(\sqrt{a}) = \dfrac{F[x]}{(x^2-a)}$ $\quad$ char $F \neq 2$

$A = (E, \sigma, b) = 4$ dim'l $\qquad$ WA $\Rightarrow M_m(D)$

$b \in F^*$

D a tw. form of $M_r(F)$

$mr = 2 \Rightarrow m=1 \;\; r=2$
$\qquad\qquad\quad\; m=2 \;\; r=1$

$b \in N_{E/F}(E^*) \longleftrightarrow [A] = 0 \qquad \longleftrightarrow \quad m=2 \Rightarrow A \simeq M_2(F)$

$b \notin N_{E/F}(E^*) \qquad \longleftrightarrow \quad m=1 \Rightarrow A$ division.

b of form $x^2 - ay^2$ $\quad$ obstruction for soln is a Brauer class!

$\mathrm{Br}(\mathbb{R}) = \{[\mathbb{R}], [\mathbb{H}]\} \simeq \mathbb{Z}/2\mathbb{Z}$

LOCAL FIELDS

F = local i.e. complete with respect to a discrete valuation, residue field is a finite field $\mathbb{F}_q$ $q = p^n$

"recall" associated to finite (separable) extensions $\ell / \mathbb{F}_q$, there are unique "unramified" extensions L/F s.t. Galois gps agree

i.e. equiv. of categories

$$\left\{ \begin{matrix} \ell \\ | \\ \mathbb{F}_q \end{matrix} \text{ finite} \right\} \longleftrightarrow \left\{ \begin{matrix} L \\ | \\ F \end{matrix} \begin{matrix} \text{finite} \\ \text{unramified} \end{matrix} \right\}$$

<u>Thm</u> every element of $Br(F)$ is equiv. to one of the form $(L, \text{frob}, \pi^\ell)$ ← uniformizer

$$[(L, \text{frob}, \pi^\ell)] = \ell[(L, \text{frob}, \pi)]$$

say, over $\mathbb{Q}_p$ $\pi = p$

get $Br(\mathbb{Q}_p) \simeq \mathbb{Q}/\mathbb{Z}$

$$[(L, \text{frob}, p)] \longmapsto 1/[L:\mathbb{Q}_p]$$

↑ is a division alg.

in fact for F local, always have

$$Br(F) = \mathbb{Q}/\mathbb{Z}$$

(via choice of uniformizer)

GLOBAL FIELDS

<u>Theorem</u> (Albert-Brauer-Hasse-Noether)

If F is a # field, Ω_F = places, then we have an exact sequence

$$0 \to Br(F) \longrightarrow \bigoplus_{v \in \Omega_F} Br(F_v) \xrightarrow{\Sigma} \mathbb{Q}/\mathbb{Z} \to 0$$

$\mathbb{Q}/\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$

$$0 \to Br(F) \longrightarrow \bigoplus_{v \in \Omega_F} (\mathbb{Q}/\mathbb{Z})_v \xrightarrow{\Sigma} \mathbb{Q}/\mathbb{Z} \to 0$$

$$(\mathbb{Q}/\mathbb{Z})_v = \begin{cases} \mathbb{Q}/\mathbb{Z} & \text{if } v \text{ discrete} \\ \mathbb{Z}/2\mathbb{Z} & v \text{ real} \\ \mathbb{Z}/\mathbb{Z} & v \text{ complex.} \end{cases}$$

$$\begin{tikzcd} Br(F) \arrow[r] \arrow[rr, bend right, "inv_v"'] & Br(F_v) \arrow[r, "\sim"] & (\mathbb{Q}/\mathbb{Z})_v \end{tikzcd}$$

Stage 3 Brauer Classes on schemes

Suppose X is a smooth projective variety over a field k.

If we are given a central simple $k(X)$-algebra A, want to know "when does A describe a family of CSA's parametrized by X?"

e.g., given a point $x \in X$ (scheme-theoretic) can ask: can we find an algebra B over $\mathcal{O}_{X,x}$ s.t. $B \otimes_{\mathcal{O}_{X,x}} k(X) \cong A$?

Language: B is an order for A over $\mathcal{O}_{X,x}$ (if f-gen)

more generally, given R, $\mathrm{frac}(R) = L$, A/L a CSA, B/R a sub R-algebra of A, module-finite as an R-module w/ $B \otimes_R L \cong A$,

we say B is an order in A.

FACT: If R is a discrete val ring, then maximal orders as above are unique up to isomorphism.

Def: We say that A/L is unramified at R if the maximal order B/R satisfies $B \otimes_R R/m$ is a CSA$/R/m$

$n = 2$

$$\mathbb{Q}_p \subseteq (L, \text{frob}, p)$$

$$\cup$$

$$\mathbb{Z}_p \; (R, \text{frob}, p)$$

R integers in L.

$R \oplus Rx \oplus Rx^2 \oplus \cdots$

$R \oplus Rx \quad x^2 = p$

$x^n = p$

$\mathbb{F}_q \oplus \mathbb{F}_q x \quad x^2 = 0$

<u>Fact</u>: (Auslander-Goldman)

If $\alpha = [A] \in Br(k(X))$ then A is unramified at every codim 1 point in X if and only if $\exists$ A' CSA / $k(X)$ $A' \sim A$ such that for each pt $x \in X$, can find a maximal order $B_x \subset A'$ ← "Azumaya Algebra"

with $B_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/m_x$ a CSA.

(i.e. A' unram. at every point)

<u>Further</u>: Brauer classes are uniquely determined by α & x.

<u>Def</u> $Br(X) = \{ \alpha \in Br(k(X)) \mid \alpha \text{ unram. at every codim 1 point} \}$

Given $X / \cancel{F}\ \mathbb{Q}$ $\quad F = \cancel{\#\text{field}}\ \mathbb{Q}$

$$\mathrm{Br}(X) \longrightarrow \mathrm{Br}(X_{\mathbb{Q}_p})$$

Compute $\alpha \in \mathrm{Br}(X)$

$$\begin{tikzcd}
X(\mathbb{Q}) \arrow[r] \arrow[dd] & \mathrm{Br}(\mathbb{Q}) \arrow[dd] \arrow[r] \arrow[rddd, bend left, "0"] & \coprod_v (\mathbb{Q}/\mathbb{Z})_v \arrow[ldd, hook'] \arrow[ddd, "\Sigma"] \\
 & & \\
X(\mathbb{A}) = \prod_{v \in \Omega} X(\mathbb{Q}_v) \arrow[r] & \prod \mathrm{Br}(\mathbb{Q}_v) & \\
 & & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

$$x \longmapsto \alpha|_x \qquad (x_v) \longmapsto (\alpha|_{x_v})_v$$

$$X(\mathbb{A})^{\alpha} = \Big\{ (x_v) \in \prod_v X(\mathbb{Q}_v) \ \Big|\ (\mathrm{inv}_v\, \alpha|_{x_v}) \in \coprod_v \mathbb{Q}/\mathbb{Z}, \text{ and } \sum_v \mathrm{inv}_v\, \alpha|_{x_v} = 0 \Big\}$$

$$X(\mathbb{A})^{\mathrm{Br}} = \bigcap_{\alpha \in \mathrm{Br}(X)} X(\mathbb{A})^{\alpha} \qquad X(\mathbb{A}) \supset X(\mathbb{A})^{\mathrm{Br}} \supset X(\mathbb{Q})$$

Language: We say that the BM obstruction is the only obstruction for rat'l pts on X if

$$X(\mathbb{A})^{\mathrm{Br}} \neq \emptyset \implies X(\mathbb{Q}) \neq \emptyset.$$

Conjecture: If $X/\mathbb{Q}$ is geometrically rational ($X_{\overline{\mathbb{Q}}}$ birat'l to $\mathbb{P}^n$)

(CT) ~~w/ Pic $X_{\overline{\mathbb{Q}}}$~~ then BM obst is only ~~o~~ obst.