Brauer - Manin, part 3: Computing the Brauer gp of varieties (over $\mathbb{Q}$).

$$\mathrm{Br}(F) = H^2(\mathrm{Gal}(F^{sep}/F), (F^{sep})^*)$$
$$= H^2(F, \mathbb{G}_m) = H^2_{\acute{e}t}(\mathrm{Spec}\, F, \mathbb{G}_m)$$

$$\mathrm{Br}(X) = H^2_{\acute{e}t}(X, \mathbb{G}_m)$$

What is étale cohomology?

$\swarrow$ Galois Cohom. $\searrow$ Analytic cohom.

$$\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(\mathbb{C}(X))$$

<u>ex:</u> if $F = \mathbb{C}$, $\quad H^2_{\acute{e}t}(X, \mathbb{G}_m) = H^2(X_{\mathbb{C}}, \mathcal{O}_X^*)$

X proj smooth var.

sheaf cohom of "units" w/r/t the standard analytic topology on the complex pts of X.

let's compute this:

$$0 \to 2\pi i \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1$$

$$H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X^*) \to H^3(X, \mathbb{Z}) \to H^3(X, \mathcal{O}_X)$$

recall: $H^2(X, \mathcal{O}_X)$ is part of $H^2(X, \mathbb{C})$

$$H^2(X, \mathbb{C}) = H^{2,0}(X, \mathbb{C}) \oplus H^{1,1}(X, \mathbb{C}) \oplus H^{0,2}(X, \mathbb{C})$$

$$H^2(X, \mathcal{O}_X) \quad H^1(X, \Omega_X) \quad H^0(X, \Lambda^2 \Omega_X)$$

$H^{1,1}$ = image of $\operatorname{Pic} X$
"easy part of Hodge conj."

$$0 \to \tfrac{1}{2} H^2(X, \mathbb{C})^{\text{trans}} \longrightarrow \operatorname{Br}(X) \to H^3(X, \mathbb{Z}) \to H^3(X, \mathcal{O}_X)$$

$$H^3(X, \mathcal{O}_X) \hookrightarrow H^3(X, \mathbb{C})$$

If not over $\mathbb{C}$, general F:

$$H^p(F, H^q(X_{F^{sep}}, \mathbb{G}_m)) \Rightarrow H^{p+q}(X, \mathbb{G}_m)$$

In the case $X/\mathbb{Q}$ smooth proj. variety, this gives:

$$0 \to \underset{\operatorname{Pic} X}{H^1(X, \mathbb{G}_m)} \to \underset{(\operatorname{Pic} X_{\bar F})^{G_F}}{H^0(F, H^1(X_{\bar F}, \mathbb{G}_m))} \to \underset{\operatorname{Br} F}{H^2(F, \mathbb{G}_m)} \longrightarrow H^2(X, \mathbb{G}_m)$$

$$\operatorname{Br} X \hookrightarrow \operatorname{Br}(F(X))$$

$$1 \to \mathcal{O}_X^* \to \mathcal{R}_X^* \longrightarrow \mathcal{R}_X^*/\mathcal{O}_X^* \to 1$$

$$\Gamma\left(\mathcal{R}_X^*/\mathcal{O}_X^*\right) = \operatorname{Div} X$$

$$\operatorname{Div} X / \Gamma(\mathcal{R}_X^*) = Cl(X) = \operatorname{Pic} X$$

$$0 \to \boxed{} \longrightarrow \frac{H^2(X, \mathbb{G}_m)}{\operatorname{im}(H^2(F, \mathbb{G}_m))} \longrightarrow H^2(X_{\bar F}, \mathbb{G}_m)^{G_F}$$

$$\ker\Big[\underbrace{H^1(F, H^1(X_{\bar F}, \mathbb{G}_m))}_{= H^1(G_F, \operatorname{Pic} X_{\bar F})} \longrightarrow \ker\big[H^3(F, \mathbb{G}_m) \to H^3(X, \mathbb{G}_m)\big]\Big]$$

Rk $\bar F \equiv F^{sep}$.

X smooth proj

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Pic}^0 X_{\bar F} \arrow[r] & \operatorname{Pic} X_{\bar F} \arrow[r] \arrow[dr] & \operatorname{Num}^1(X_{\bar F}) \arrow[r, hook] & H^2(X_{\bar F}, \mathbb{C}) \\
& & & NS(X_{\bar F}) \arrow[u, "\simeq \text{ up to torsion}"'] &
\end{tikzcd}$$

$\operatorname{Pic}^0 X_{\bar F}$ = ab. variety

$NS(X_{\bar F})$ alg. equiv.

$NS(X_{\bar F})$ = f.g. ab gp.

<u>FACT</u>: if $X_{\bar F}$ is rat'l then $\mathrm{Pic}^0 X_{\bar F} = 0$

<u>Case</u> $\dim X = 2$

$\exists$ "Classification of surfaces:"

if X is geometrically rat'l, then can find birat'l model

X' s.t. $-K_{X'}$ ample

$\Rightarrow$ can choose X' to be "del Pezzo"

WLOG $\quad X_{\bar F} = Bl_{p_1, \dots, p_r} \mathbb{P}^2 \qquad r \leq 8$

nice thing here: Picard gps well studied,

can write down...

Want to understand $\quad H^1(G_F, \mathrm{Pic}\, X_{\bar F})$

$$X_{\bar F} = \mathbb{P}^2_{\bar F} \quad \Rightarrow \quad \mathrm{Pic}\, X_{\bar F} = \mathbb{Z}$$

$G_F \circlearrowright \mathbb{Z}$ ampleness preserved

$\Rightarrow$ trivial action

$$H^1(G_F, \mathbb{Z}) = \mathrm{Hom}_c(G_F, \mathbb{Z}) = 0$$

$$\mathrm{Hom}(\mathrm{Gal}(E/F), \mathbb{Z}) = 0$$

$X = x^3 - ay^3 - a^2z^3 + a\,xyz = b$?

$\sim N$ on 3 dim'l subspace of $(a,b)_3$

$Bl_p \mathbb{P}^2_F = X_F$ $\qquad$ $\mathrm{Pic}\, X_F = \mathbb{Z}H \oplus \mathbb{Z}E$

$G_F \circlearrowright \mathrm{Pic}\, X_F$

$E^2 = -1$ $\qquad$ $(aH + bE)^2 = a^2 - b^2 = -1$

$\Leftarrow$ $(a-b)(a+b) = -1$

E fixed by G_F

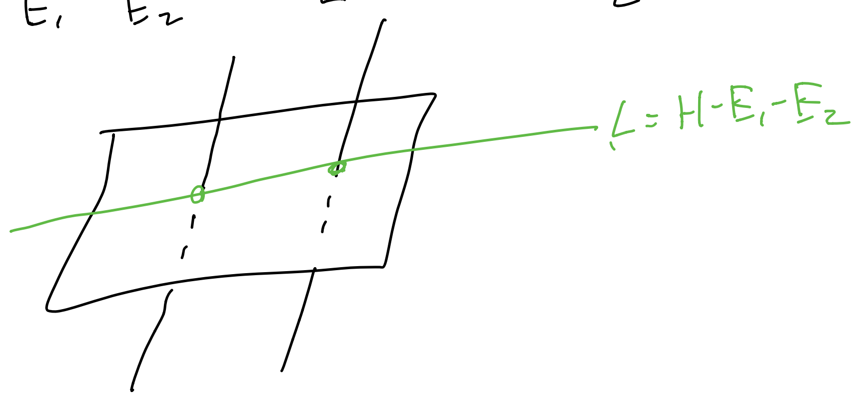
so is H. $\qquad$ boring.

$Bl_{p_1,p_2} \mathbb{P}^2 = X_F$

$H \quad E_1 \quad E_2 \qquad L = H - E_1 - E_2$

$G_F : H \to H$

$E_i \to E_j$



$$H^1(G_F, \operatorname{Pic} X_{\bar{F}}) = H^1(G_F, \mathbb{Z}H \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2)$$

$$= \underset{\substack{\shortparallel \\ 0}}{H^1(G_F, \mathbb{Z}H)} \times H^1(G_F, \mathbb{Z}E_1 \times \mathbb{Z}E_2)$$

action comes from

$$G_F \longrightarrow \operatorname{Aut}(\cdot, \cdot)$$

$$G_F \longrightarrow \mathbb{Z}/2\mathbb{Z} \quad \text{if nontriv.}$$

$\Downarrow$

L/F quadratic.