Lecture 3: Algebraic closures

Tuesday, January 24, 2017 9:30 AM

Def A poly $f \in F[x]$ splits if it factor into linr factors

Def We say E/F is a splitting field for $f \in F[x]$ if f splits in E and if E is generated by the roots of F
(i.e. $E = F(\alpha_1, \dots, \alpha_n)$, $\alpha_i \in E$ are roots of $f(x)$.)

Lem splitting fields exist

Pf: If $f(x) \in F[x]$ is a poly, let $p(x)$ be irred factor, consider $F[x]/p(x) = E \ni [x] = \alpha_1$ is a root of $p(x)$, factor this out, get $f = gp$ g smaller deg / $E = F(\alpha_1)$
do this w/ g, continue, eventually get an ext which is gen by roots by construction, and in which f factors completely $\square$.

Def: a map of field extensions $\psi: E_1/F_1 \longrightarrow E_2/F_2$ is a ring hom $E_1 \xrightarrow{\psi} E_2$ s.t. $\psi(F_1) \subset F_2$

If $\varphi: F_1 \to F_2$ given and if $\psi|_{F_1} = \varphi$ then we say that ψ is a φ-map, $\psi: E_1/F_1 \xrightarrow[\varphi]{} E_2/F_2$

if $F_1 = F_2 = F$, $\varphi = id_F$ then if ψ is a φ-map we also say ψ is an F-map $(=$ F-algebra map$)$

<u>Lem</u> If $E_1 = F_1(\alpha)$ algebraic, $\varphi: F_1 \longrightarrow F_2$, then $\mathrm{Hom}_\varphi(E_1/F_1, E_2/F_2)$ is in bijection w/ roots of $\varphi m_{\alpha,F_1}$ in E_2

<u>Pf:</u> $E_1 \cong F_1[x]/_{m_{\alpha,F_1}} \xrightarrow{\psi} E_2$

$$F_1 \xrightarrow{\varphi = \psi|_{F_1}} F_2$$

$$m_{\alpha,F_1} \longrightarrow \varphi m_{\alpha,F_1} = 0$$

ψ determined by $\psi(x) = \alpha \in E_2$, α can be chosen freely satisfying $\varphi m_{\alpha,F_1}(x) \rightsquigarrow \varphi m_{\alpha,F_1}(\alpha) = 0$ Δ.

<u>Algebraic Closures</u>

<u>Relative version:</u>

<u>Def</u> if E/F a field extension, then the algebraic closure of F in E is $\{\alpha \in E \mid \alpha \text{ algebraic over } F\}$

<u>Remark/Lem</u> this is a subfield since if $\alpha, \beta \in E$ algebraic

then $[F(\alpha,\beta) : F(\beta)] =$ deg of min poly of α over $F(\beta)$ = so so this finite

simple alg ext.

& $[F(\beta):F] < \infty \Rightarrow [F(\alpha,\beta):F]$ finite

$\Rightarrow$ D.

Def F is algebraically closed if every poly $f \in F[x]$ splits in F.

Def E/F is an algebraic closure if E is algebraically closed & E/F is algebraic.

As we'll see, $\mathbb{C}$ is not an alg. closure of $\mathbb{Q}$.

Lem TFAE for E/F algebraic

1) E alg closed
2) E an alg. closure
3) $\nexists$ alg. exts of E
4) $\nexists$ exts. L/E s.t. L/F algebraic
5) every poly in E has a root in E.

Lem if $F \subset L$ L algebraically closed then the algebraic closure of F in L is an algebraic closure of F.

Thm Algebraic closures exist.

Lem E/L alg & L/F alg $\Rightarrow E/F$ alg.

Lem If E/F is algebraic, then $|E| \le |F[x]| = \aleph_0 |F|$

Pf: Consider pairs $P = \{(\alpha, f) \mid \alpha \in E, f \in F[x], f(\alpha) = 0\}$

$P \twoheadrightarrow E$ since E is algebraic. $\Rightarrow |P| \geq |E|$

$\psi : P \longrightarrow F[x]$, inverse images $\psi^{-1}(f)$ is finite.

$\Rightarrow |P| \leq |F[x]| \aleph_0 = |F[x]|$ $\quad |F[x]| \geq |P| \geq |E|$.

$\square$.

Pf of thm

Choose a set S, disjoint from F s.t. $|S| > |F[x]|$ (e.g. $S = 2^{F[x]}$)

Consider the set $\mathcal{F}$ of all algebraic field extensions E of F s.t.
the underlying set of E is a subset of $F \cup S$.

Note, if L/E is algebraic, $E \in \mathcal{F}$ then $|L| < |S|$ and so
$|L \setminus E| < |S \setminus E|$ so can find an injection of sets $L \setminus E \overset{i}{\hookrightarrow} S \setminus E$
can use this to define a field structure on $E \cup i(L \setminus E) = K$
s.t. this field K is E isom to L.

Now, if E_λ $\lambda \in \Lambda$ is a totally ordered (by inclusion)
collection of field exts in $\mathcal{F}$, then $\bigcup_\Lambda E_\lambda$ is also in $\mathcal{F}$

$\Rightarrow$ Zorn $\exists$ max'l elmts in $\mathcal{F}$.

Let $E \in \mathcal{F}$ max'l. then if E'/E algebraic
then $E' \simeq_E K \in \mathcal{F}$ and E isom $\Rightarrow K \supseteq E \Rightarrow$ max'l $\Rightarrow K = E$

$\Rightarrow E' = E \Rightarrow$ E alg closed

$\square$.