Profinite groups

If $\mathcal{C}$ is a category, $\mathbb{D}$ = diagram (partially ordered set) considered as a category.

$$d \le d' \iff d \to d'$$

$\mathbb{D} \xrightarrow{F} \mathcal{C}$ functor (a diagram in $\mathcal{C}$)

then can take $\varinjlim\limits_{d \in \mathbb{D}} F(d)$ $\qquad \varprojlim F(d)$

i.e.

$$\begin{tikzcd}
 & c_i \arrow[r] \arrow[dr] \arrow[rr, bend left] & c_k \arrow[r] & \varinjlim c_\ell \\
\varprojlim c_\ell \arrow[ur] \arrow[urr, bend left] \arrow[rr] & & c_j \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Def We say that $a = \varinjlim a_i$ (if given a_i's $\overset{F(i)}{=}$ in some diagram as above)

if we are given morphisms $a_i \xrightarrow{\varphi_i} a$

s.t. for all $i \le j$

$$\begin{tikzcd}
a_i \arrow[rr, "\varphi_i"] \arrow[dr] & & a \\
 & a_j \arrow[ur, "\varphi_j"'] &
\end{tikzcd}$$

commutes &,

if we have any other object b w/ morphisms $\psi_i : a_i \to b$ s.t.

$$\begin{tikzcd}
a_i \arrow[rr] \arrow[dr] & & b \\
 & a_j \arrow[ur] &
\end{tikzcd}$$

then there is a unique $a \to b$ s.t. for all i, the diagram

$$\begin{tikzcd}
a_i \arrow[rr] \arrow[dr] & & b \\
 & a \arrow[ur] &
\end{tikzcd}$$

commutes.

ex:

$$\begin{tikzcd} a_1 \arrow[dr, hook] & \\ & a = a_1 \sqcup a_2 \\ a_2 \arrow[ur, hook] & \end{tikzcd} \qquad \mathcal{D} = \begin{tikzcd} \bullet \arrow[loop right] \\ \bullet \arrow[loop right] \end{tikzcd}$$

ex: $\mathcal{C} = $ Sets
$\sqcup$ = disjoint union.

$\mathcal{C} = $ Ab gps
$\sqcup = \oplus$

$\mathcal{C} = $ groups
$\sqcup = *$

$\mathcal{C} = $ Rings (commutative)
$\sqcup = \otimes_{\mathbb{Z}}$

$a_1 \hookrightarrow a_2 \hookrightarrow a_3 \hookrightarrow a_4 - - -$

answer in all above settings is $\bigcup a_i$

Def $\mathcal{D}$ is filtered if it has upper bounds of pairs of elmts
(inductive, directed) in poset case

Dually: we write $\varprojlim a_i = a$

$$\begin{tikzcd} & a_i \arrow[d] \\ a \arrow[ur] \arrow[r] & a_j \end{tikzcd}$$

if $\mathcal{D}$ has common lower bounds $\Rightarrow$ $\mathcal{D}$ is cofiltered, projective

$$\begin{tikzcd} & a_1 \\ a_1 \prod a_2 = a \arrow[ur] \arrow[dr] & \\ & a_2 \end{tikzcd}$$

General language:

If $\mathcal{C}' \subset \mathcal{C}$ subcategory, we say that $a \in \mathcal{C}$ is pro-$\mathcal{C}'$ if it is given as an $\varprojlim$ of a projective system in $\mathcal{C}'$.

$\mathcal{C}$ = groups $\mathcal{C}'$ = finite groups pro-finite groups.

I p.ordered index set w/ lower bounds

gps G_i

$$\begin{tikzcd} & & G_j \arrow[d] \\ \varprojlim G_i \arrow[urr] \arrow[rr] & & G_k \end{tikzcd}$$

Construction (for groups)

Given a system $\{G_i\}_{i\in I}$, $\{\varphi_{ij}\}_{i\le j}$ as above, define

$$\varprojlim G_i = \{(g_i)_{i\in I} \mid g_i \mapsto g_j \text{ in map } G_i \to G_j,\ i\le j\} \subset \prod G_i$$

comes w/ natural projection $\varprojlim G_i \to G_j$

universal: if $H \to G_i$'s, then $H \to \prod G_i$

and lands in $\varprojlim G_i$

Remark: by proper abuse $\varprojlim G_i = \varprojlim G_i$

If $G = \varprojlim G_i$ then let $N_i = \{g \in G \mid \varphi_i g = 1\}$

$G \xrightarrow{\varphi_i} G_i$

the hN_i form a ~~sub~~basis for a topology "Krull top"

Observation: If E/F is an algebraic extension,

then $E = \varinjlim_{\substack{L \subset E \\ \text{finite}/F}} L$

let $\underset{\text{Aut}}{\text{Gal}}(E/F) = \varprojlim_{\substack{(L\subset E)^{op} \\ \text{Galois}/F}} \overset{\text{Aut}}{\text{Gal}}(L/F)$

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ex: $\begin{array}{c} L \\ \big|\, \mathbb{Z}/n \\ \mathbb{F}_p \end{array}$ $\qquad \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) = \varprojlim \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}$

$$\uparrow$$

$$\text{Aut}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$$

If E/F algebraic we say Galois if

- normal separable
- $F = E^{\text{Gal}(E/F)}$

if norm, sep, let $x \in E^{\text{Gal}(E/F)}$, say $x \in L/F$ finite

$\Rightarrow$ norm closure of L also in E WLOG L norm, sep.

know that if $x \in L^{\text{Gal}(L/F)}$ then $x \in F$.

if $\sigma \in \text{Gal}(L/F)$, why is $\sigma x = x$?

ETS: $\mathrm{Gal}(E/F) \twoheadrightarrow \mathrm{Gal}(L/F)$.

Consider K/L s.t. K/F Galois.

note, all fields $F \subset K' \subset E$ are contained in one of these.

Let $\mathcal{C} = \{$ chains $K_1 \subset K_2 \subset \cdots$ linearly ordered, infinite? $\}$

let $c_1 \le c_2$ if for each $K_i \le K'_j$ some j.

$c_1 = \{K_i \subset \cdots\}$, $c_2 = \{K'_i \cdots\}$

note chains of chains have upper bounds. by diagonals.

$\implies \exists$ maximal one.

linearly ordered, union is E.

if $F = E^{\mathrm{Gal}(E/F)}$, $F \subset L \subset E$ finite,

then finitely many conjugates of L in E

so say

$$\begin{array}{c} \tilde{L} \\ | \\ L \\ | \\ F \end{array}$$

) fixed by $\mathrm{Gal}(E/F)$ if $x \in \tilde{L}$, and fixed by $\mathrm{Gal}(\tilde{L}/F)$

$\Rightarrow$ fixed by $Gal(E/F) \Rightarrow x \in F$.

Fundamental theorem of Gal thy

If E/F is G-Galois, then there is an inclusion reversing bijection between

- subextensions L/F
- closed subgroups $H < G$.

Definition: We say that a topological space X is a classifying space for a finite group G if $\pi_1 X \simeq G$ and if $\tilde{X} \longrightarrow X$ is the universal cover, then $\tilde{X}$ is contractible.

$$\begin{tikzcd} \tilde{X} \arrow[r, shift left, "f"] & pt \arrow[l, shift left, "g"] \end{tikzcd}$$

i.e. a homotopy $gf \simeq 1_{\tilde{X}}$

i.e. $\exists\ [0,1] \times \tilde{X} \xrightarrow{h} \tilde{X}$

$h(0,x) = x$ all x

$h(1,x) = p$ some fixed pt $\tilde{X}$.

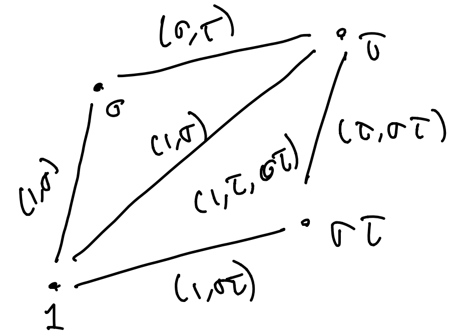
Fact: these exist, unique up to homotopy equivalence.

$BG \qquad K(G,1)$ (G Abelian)

Construction: make a contractible space EG on which G acts without stabilizers, set $BG = EG/G$.

if A is an Abelian gp $\quad H^n(G,A) = \underset{\text{singular.}}{H^n(BG,A)}$

Def of EG:



EG: 0 cells G

1 cells $G\times G$

2 cells $G\times G\times G$.

BG: 0 cell $G\backslash G$

1 cells $G\backslash G\times G$

i-cells $i+1$ tuples $(g_0,g_1,\dots,g_i)/\sim$

$[g_0,\dots,g_i]$ = class $(tg_0,\dots,tg_i)\sim(g_0,\dots,g_i)$, $t\in G$.

$$\partial_j[g_0,\dots,g_i] = [g_0,\dots,\hat{g}_j,\dots,g_i]$$

$$[g_0,\dots,g_i] \longrightarrow [1,h_1,h_2,\dots,h_i]$$

$$\partial_j[1,h_1,\dots,h_i] = \begin{cases} [1,\dots,\hat{h}_j,\dots] & j\neq 0 \\ [1,h_1^{-1}h_2,\dots] & \end{cases}$$

$[h_1,\dots,h_i]$

Alternate def:

If G a profinite group, we say that A is a continuous G-mod if action is continuous

topological similar..

$$G \times A \xrightarrow{act} A$$ (unless otherwise specified, assume A has discrete top)

$$G \times A \xrightarrow{act, \pi_2} A \times A$$ stabilizers open.

<u>Def:</u> $H^n_c(G, A) = \varinjlim_{N \text{ open}} H^n(G/N, A^N)$

if G finite $H^n(G, A) \equiv Ext^n_G(\mathbb{Z}, A)$

$$H^0(G,A) = Hom_G(\mathbb{Z}, A) = A^G$$

$Ext^1(\mathbb{Z}, A)$

$$0 \to I_G \to \mathbb{Z}G \xrightarrow{N} \mathbb{Z} \to 0$$

I_G = gen by $\langle \sigma - 1 \rangle$, $\sigma \mapsto 1$

A a, $\sigma - 1 \to \sigma a - a$

$(\sigma - 1) \to b_\sigma$

$$0 \to Hom_G(\mathbb{Z}, A) \to Hom_G(\mathbb{Z}G, A) \to Hom_G(I_G, A)$$

$Hom_G(\mathbb{Z}G, A) = A$

$$\to Ext^1_G(\mathbb{Z}, A) \to Ext^1_G(\mathbb{Z}G, A)$$

$Ext^1_G(\mathbb{Z}, A) = H^1(G, A)$, $Ext^1_G(\mathbb{Z}G, A) = 0$

$$(\sigma-1)+\sigma(\tau-1) = \sigma-1+\sigma\tau-\sigma = \sigma\tau-1$$

$$b_\sigma + \sigma(b_\tau) = b_{\sigma\tau}$$

Def of $\mathrm{Ext}^i_G(M,N)$ (Ab gps)

universal s.t.

given any s.e.s. of G-mods $0 \to A \to B \to C \to 0$

& any G-mod M, get l.e.sequences

$$0 \to \mathrm{Ext}^0_G(M,A) \to \mathrm{Ext}^0_G(M,B) \to \mathrm{Ext}^0_G(M,C) \to \mathrm{Ext}^1_G(M,A) \to \cdots \to \mathrm{Ext}^2 \cdots$$

$$0 \to \mathrm{Ext}^0_G(C,M) \to \mathrm{Ext}^0_G(B,M) \to \cdots$$

$\mathrm{Ext}^0_G(M,N) = \mathrm{Hom}_G(M,N)$ & $\mathrm{Ext}^i_G(M,N) = 0$

if $i>0$ M, proj or

N injective.

<u>Construction</u>: Given M, let

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

be a projective resolution of G-modules

$$\mathrm{Ext}^i_G(M,N) = \frac{\ker\left(\mathrm{Hom}_G(P_i,N) \to \mathrm{Hom}_G(P_{i+1},N)\right)}{\mathrm{im}\left(\mathrm{Hom}_G(P_{i-1},N) \to \mathrm{Hom}_G(P_i,N)\right)}$$