Question: • what is special about Gal. gps as groups?
(profinite)

• what is special about Gal. cohomology as gp cohom. of profinite gps?

($\Rightarrow$ and what do these answers tell us about arithmetic, algebra, geometry?)

example: Birational anabelian geometry

idea: given X: solns to system of alg. eqns / $\mathbb{C}$
(irreducible / alg. var.)

cut out by functions $f_1, \ldots, f_n \in \mathbb{C}[\vec{x}]$

consider $\dfrac{\mathbb{C}[x_1, \ldots, x_n]}{(f_1, \ldots, f_n)}$ "the ring of regular functs on X"

F = fractions of $\uparrow$

Conj? (if conj $\Rightarrow$ Groth's conj)
F determined by $\mathrm{Gal}(\bar{F}/F)$

Recall: $H^n(G, A) = H^n_{sing}(BG, A)$
trivial G-action

$G \longleftrightarrow$ top space

Gp cohom ⇝ coh. gps of top spaces.

Practice computations

$$H^n(F, A) = H^n_c(\mathrm{Gal}_F, A)$$

$\mathbb{G}_m$ = multiplicative gp

$\mathbb{G}_a$ = additive group

$$H^n(F, \mathbb{G}_m) = H^n_c(\mathrm{Gal}(F^{sep}/F), (F^{sep})^*)$$

$$H^n(F, \mathbb{G}_a) = H^n_c(\mathrm{Gal}(F^{sep}/F), F^{sep})$$

$H^1(F, \mathbb{G}_m)$ classifies forms of a 1-dim'l vector space.

X = alg. structure, $A = \mathrm{Aut}(X \otimes E)$ E/F G-Galois

$H^1(G, A) \longleftrightarrow$ Iso classes of X'/F s.t. $X' \otimes E \cong X \otimes E$

$H^1(F, \mathbb{G}_m) = 0$

$$1 \to \mu_\ell \to (F^{sep})^* \xrightarrow{\cdot \ell} (F^{sep})^* \to 1$$

char $F \nmid \ell$

ℓ^{th} roots of 1.

$$1 \to \mu_\ell \to \mathbb{G}_m \to \mathbb{G}_m \to 1$$

apply Gal. cohom:

$$1 \to H^0(F,\mu_\ell) \to H^0(F,\mathbb{G}_m) \xrightarrow{\text{mult. by } \ell} H^0(F,\mathbb{G}_m) \to H^1(F,\mu_\ell) \to H^1(F,\mathbb{G}_m)$$

$$H^1(F,\mathbb{G}_m) = 0$$

$$H^0(G,A) = A^G \qquad H^0(F,\mathbb{G}_m) = \left((F^{sep})^*\right)^{Gal(F^{sep}/F)} = F^*$$

$$F^* \xrightarrow{\cdot \ell} F^* \to H^1(F,\mu_\ell) \to 0$$

$$\boxed{H^1(F,\mu_\ell) = F^*/(F^*)^\ell}$$

<u>ex:</u> if $\mu_\ell \subset F \Rightarrow$ Gal action on μ_ℓ is trivial

$$\mu_\ell \cong \mathbb{Z}/\ell\mathbb{Z}$$

$$\Rightarrow \underset{F^*/(F^*)^\ell}{H^1(F,\mu_\ell)} = H^1(F,\mathbb{Z}/\ell\mathbb{Z}) = \mathrm{Hom}(G_F,\mathbb{Z}/\ell\mathbb{Z})$$

"cyclic Galois exts of F"

"Kummer theory"

How to see cyclic extensions, when $\mathbb{Z}/\ell\mathbb{Z} \cong \mu_\ell$

<u>ASIDE</u>: Given a finite gp G, how to construct <u>all</u> G-Galois exts.

Choose a nice action of G on $k[x_1, \ldots, x_n]$ (acting linearly & faithfully on coords),

$E = k(x_1, \ldots, x_n)$, $F = E^G$

$$\begin{array}{c} E \\ \mid G \\ F \end{array}$$

<u>magic</u>: if

$$\begin{tikzcd} K \arrow[d, dash, "G"] & \\ L \arrow[r, dash] & k \end{tikzcd}$$

any G-Gal ext,

then $\begin{array}{c} K \\ \mid G \\ L \end{array}$ arises from specialization of the x_i's.

If Lucky: $F = k(s_1, \ldots, s_r)$

If $\mu_\ell \subset F$, ρ a prim ℓ^{th} root of 1,

$\mathbb{Z}/\ell$ action $F(x)$ $\quad \underset{\langle\sigma\rangle}{\underset{\shortparallel}{\mathbb{Z}/\ell\mathbb{Z}}} \circlearrowright F(x) \quad \sigma(x) = \rho x$

exercise $F(x)^{\mathbb{Z}/\ell\mathbb{Z}} = F(y) \quad y = x^\ell$

$$\begin{array}{l} F(x) = F(y)[x]/x^\ell - y \\ \quad | \\ F(y) \end{array}$$

$\Rightarrow$ every $\mathbb{Z}/\ell$ extension has form $F[x]/x^\ell - a$ (specialize $y \mapsto a$)

$H^1(F, \mathbb{G}_a) = 0$

$$0 \to \mathbb{F}_p \to \underset{F^{sep}}{\mathbb{G}_a} \xrightarrow{x^p - x} \underset{F^{sep}}{\mathbb{G}_a} \to 0$$

β

$$(x+y)^p - (x+y) = x^p + y^p - x - y$$
$$= x^p - x + y^p - y$$

$x^p - x = 0$

$x = 0, 1, 2, \ldots, p-1$

roots of $f: x^p - x - a$ ---> is this separable

$f' = -1 \Rightarrow$ no repeated roots.

long exact seq:

$$0 \to H^0(F, \mathbb{F}_p) \to H^0(F, \mathbb{G}_a) \to H^0(F, \mathbb{G}_a) \to H^1(F, \mathbb{F}_p)$$

$$H^0(F,\mathbb{G}_a) = (F^{sep})^{Gal(F^{sep}/F)} = F$$

$$H^1(F, \mathbb{G}_a) = 0$$

Artin-Schreier map

$$F \xrightarrow{\wp} F \to H^1(F, \mathbb{F}_p) \to 0$$

$$H^1(F, \mathbb{F}_p) = F/\wp(F)$$

$F(x)$

$\mathbb{F}_p \circlearrowright F(x)$

$i \mapsto (x \mapsto x+i)$

$y = x^p - x$

$$F(x) = \frac{F(y)[x]}{x^p - x - y}$$

$$\begin{array}{c} F(x) \\ \big\downarrow \mathbb{F}_p \\ F(y) \end{array}$$

Artin-Schreier: all cyclic exts of F (char p) are of the form $F[x]/x^p - x - a$, $a \in F$.

Inflation & Restriction (Group cohomology)

Universal property of gp coh:

Finite gp G, G-module A

$H^0(G,A) = A^G$

& if $0 \to A \to B \to C \to 0$ s.e.s. of G-mods,

get l.e.s. of gps (functorially "δ-functor")

$$\to H^{i-1}(G,C) \to H^i(G,A) \to H^i(G,B) \to H^i(G,C) \to H^{i+1}(G,A) \to \cdots$$

$H^i(G,A) = 0$ if A is (a projective $\mathbb{Z}G$-module) $\mathbb{Z}G$.
$i > 0$.

ex: $H^i(C_n, \mathbb{Z})$ $\quad C_n = \langle \sigma \rangle$

$$\mathbb{Z}C_n \longrightarrow \mathbb{Z}C_n \xrightarrow{N} \mathbb{Z} \to 0$$

$$1 \longmapsto (\sigma - 1) \qquad \sigma^i \longmapsto 1$$

$$0 \to I_{C_n} \to \mathbb{Z}C_n \to \mathbb{Z} \to 0$$

$\quad\; \overset{\shortparallel}{\ker N}$

$$\begin{tikzcd} \to H^1(C_n, \mathbb{Z}C_n) \arrow[r] & H^1(C_n, \mathbb{Z}) \arrow[r] & H^2(I_{C_n}) \arrow[d] \\ & & H^2(\mathbb{Z}C_n) \end{tikzcd}$$

0 0 0

0

Restriction: Given $H < G$,

$H^n(G, A) \longrightarrow H^n(H, A)$ (restrict cocycles to H)

$n=0$

$A^G \longrightarrow A^H$ inclusion

both δ-functors on G-mods

$A^G \to A^H$ induces a morphism of δ-functors

Brief aside on δ functors:

δ: Functors

sequence of functors $\delta^0, \delta^1, \delta^2, \ldots : G\text{-mod} \to Ab$

$\delta^i(A) = $ ab. gp.

such that s.e.s. of G-mods $\rightsquigarrow$ l.e.s. of Ab. gps.

assumption: $\delta^i(\mathbb{Z}G) = 0$ if $i > 0$ (effacability)

Main things: 1. δ is "uniquely determined" by δ^0 up to unique isom. of δ-functors and 2. $\mathrm{Hom}(\delta, \delta') = \mathrm{Hom}(\delta^0, \delta'^0)$

where $\mathrm{Hom}(\delta^0, \delta'^0)$ = Nat. transformations from δ_0 to δ_0'

$\mathrm{Hom}(\delta, \delta')$ = A collection of nat. trans. $\delta_i \to \delta_i'$ s.t. maps between l.e.s. have a bunch of commuting squares

for each A, have a morphism $\delta_0 A \to \delta_0' A$ natural, in the sense that

$$\begin{array}{ccc} \delta_0 A & \longrightarrow & \delta_0' A \\ \downarrow & & \downarrow \\ \delta_0 B & \longrightarrow & \delta_0' B \end{array}$$

commutes whenever have $A \to B$

ex: observe that $H^i(G, A) = \delta^i$ $H^i(H, A) = \delta'^i$ are both δ-functors

$\mathbb{Z}G = \bigoplus \mathbb{Z}H$ as H-mods.

$\quad \shortparallel$

$\bigoplus \mathbb{Z}Hg_i$

g_i right coset reps

$H^i(H, \mathbb{Z}G) = H^i(H, \bigoplus \mathbb{Z}H)$

$= \bigoplus H^i(H, \mathbb{Z}H) = 0$

(more generally $H^i(H, A \oplus B)$

$\shortparallel$

$H^i(H, A) \oplus H^i(H, B)$)

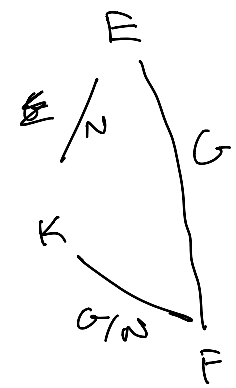
Inflation:

$\bar{G} = G/N \qquad N \triangleleft G.$

A G-module

A^N is a $G/N = \bar{G}$ module.

$$H^n(G/N, A^N) \xrightarrow{\text{inf}} H^n(G, A)$$



$$\boxed{\begin{tikzcd} c: (G/N)^n \arrow[r] & A^N \arrow[d] \\ G^n \arrow[u] & A \end{tikzcd}}$$

$$H^0(G/N, A^N) \to H^0(G, A)$$

$$\underset{\overset{\shortparallel}{A^G}}{(A^N)^{G/N}} \to A^G \qquad \text{(composition of } \delta\text{-functors)}$$

"edge map"

$$H^n(F, A) = \varinjlim_{\substack{N \triangleleft G_F \\ \text{open}}} H^n(G_F/N, A^N) \quad \text{(limit via inflation)}$$

Cup product:

$$H^n(G, A) \times H^m(G, B) \xrightarrow{\cup} H^{n+m}(G, A \otimes_{\mathbb{Z}} B)$$

$$A^G \times B^G \longrightarrow (A \otimes B)^G$$

$$(a, b) \longmapsto (a \otimes b)$$

induces

$$H^n(F,A)\times H^m(F,B) \xrightarrow{\cup} H^{n+m}(F, A\otimes_{\mathbb{Z}} B)$$

if A is a ring: $H^n(F,A)\times H^m(F,A)$

$$\to H^{n+m}(F, A\otimes A)$$

$$\to H^{n+m}(F,A)$$

Based on circumstantial evidence, interesting gp coh. aren't $H^n(F,\mathbb{Z}/\ell\mathbb{Z})$ but rather close to

$$H^n(F, \underbrace{\mu_\ell\otimes\mu_\ell\otimes\cdots\otimes\mu_\ell}_{n\text{-times}}) = H^n(F,\mu_\ell^{\otimes n})$$

$$H^n(F,\mu_\ell^{\otimes n-1})$$

$$H^*(F,\mu_\ell^*) = \bigoplus H^n(F,\mu_\ell^{\otimes n}) \text{ \underline{is a ring.}}$$

$$H^0 = \mathbb{Z}/\ell\mathbb{Z}$$

__Conjecture__ (Bloch-Kato) / Norm-Residue Isomorphism theorem (Voev., Weibel-Suslin, Rost, Voevodsky)

if ℓ not divisible by char F, then $H^*(F,\mu_\ell^*)$ is generated in degree 1, with relations in deg 2.

Concretely: $H^*(F, \mu_\ell^{\otimes *}) = \underbrace{\dfrac{\mathbb{Z}/\ell\mathbb{Z}\langle \bar{a} \rangle_{\bar{a} \in F^*/F^{*\ell}}}{\bar{a}\cdot\bar{b} = 0 \text{ if } a+b=1.}}_{K_*^M(F)/\ell}$