Analogy

$$\begin{tikzcd} TS^2 \arrow[r, leftrightarrow] \arrow[d] & B = \text{unit } S^1\text{-bundle.} \arrow[dl, "\pi"] \\ S^2 \arrow[ur, dashed, bend right, "?"'] & \end{tikzcd}$$

$\leftarrow$ can't exist.

section amounts to a choice of unit tangent vector at each pt in S^2 in a continuous way. $\Rightarrow$ comes from global geometry of S^2.

Given $\begin{tikzcd} X \arrow[d, "\pi"] \\ S \end{tikzcd}$, can we find a section $\begin{tikzcd} X \arrow[d] \\ S \arrow[u, dashed, bend right, "s"'] \end{tikzcd}$?

one way to get an obstruction: choose $\alpha \in H^n(X, A)$ s.t. $H^n(S, A) = 0$ show that $s^*\alpha \neq 0$.

For us, use $\alpha \in H^2(X, \mathbb{G}_m) \subset H^2(k(X), \mathbb{G}_m)$

$k = \mathbb{Q}$ $\quad X/k$ variety $\quad k(X)$ f.field

given a rat'l pt $x \in X(L)$, can restrict $\alpha|_x \in H^2(L, \mathbb{G}_m)$

in particular $\begin{tikzcd} \mathbb{Q}_p & & \mathbb{R} \\ & \mathbb{Q} \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$ strategy will be:

determine for each $x \in X(\mathbb{Q}_p)$, $\alpha|_x \in H^2(\mathbb{Q}_p, \mathbb{G}_m)$

magic: if X is nice enough, can say what $\alpha|_x$ is w/out knowing exactly what x is.

show if $\beta \in H^2(\mathbb{Q}, \mathbb{G}_m)$, its restrictions to $\mathbb{Q}_p, \mathbb{R}$ must satisfy same compatibilities.

Define $X \subset \mathbb{P}^4$ $[s;t;x;y;z]$ $[0; \frac{6}{\sqrt{-15}} : -5 : 1 : 1]$

$s^2 = xy + 5z^2 \qquad s^2 - 5t^2 = x^2 + 3xy + 2y^2$

$[0; 1 : 1 : 0 : (\sqrt{-5})^{-1}]$

Part 1: $X(\mathbb{Q}_p) \neq \emptyset$ all p, $X(\mathbb{R}) \neq \emptyset$.

Observation: $\left(\frac{-1}{p}\right)\left(\frac{-5}{p}\right) = \left(\frac{5}{p}\right)$

p odd

one of these is 1 $\Rightarrow$ for each p including $\mathbb{R}$

either -1 or 5 or -5 is a square $[1:1:1:0:i]$

$\Rightarrow$ get $[1 : i : 1 : 1 : 0]$ on $X(\mathbb{Z}/p\mathbb{Z})$

$[0:0:\sqrt{5}:1:i]$

$[0 : 1 : \sqrt{-5} : 0 : 0]$ $\qquad [\sqrt{5} : 1 : 0 : 0 : 1]$

$[\sqrt{-5} : 0 : 0 : 0 : 1]$

$s^2 = 5z^2$ $\qquad$ $s^2 = 5t^2$ $\qquad$ $(x = y = z)$

$?$ $\qquad$ $?$

1 $\qquad$ 1

$X^2 - 5$

$2X$

$p = 2$, use that $\sqrt{-15} \in \mathbb{Q}_2$ $\qquad$ $X^2 + 15$

$[-5 : 1 : 0 : 1 : \frac{6}{\sqrt{-15}}]$? $\qquad$ 17 mod 32

$X \rightsquigarrow X - \frac{f(x)}{f'(x)}$

$[0 : \frac{6}{\sqrt{-85}} : -5 : 1 : 1]$

$X^2 + 15$ $\qquad$ $2(17)$

$(17)^2 + 15 = 304 = 16 \cdot 19$

Consider the following (quaternion) algebra

$\left(\frac{a,b}{F}\right)_2 = (a,b)$ i, j

char $F \neq 2$ $a,b \neq 0$ $i^2 = a$ $j^2 = b$ $ij = -ji$

basis $1, i, j, ij$

also: can describe this as:

$(F(i), \sigma, b) = (a,b)$

$$\begin{array}{ccc} \text{"}F(\sqrt{a})\text{"} & = & F[x]/x^2-a \\ \cup\!\!\uparrow & & \uparrow j \\ F(i)X_1 & \oplus & F(i)X_\sigma \end{array}$$

$X_\tau X_\gamma = c(\tau,\gamma) X_{\tau\gamma}$

$X_\tau \lambda = \tau(\lambda) X_\tau$

$c(1,1) = 1$
$c(1,\sigma) = c(\sigma,1) = 1$
$c(\sigma,\sigma) = b$

Consider the algebra

$A = (5, (x+y)/x)$ over $k(X)$

$X: \; s^2 = xy + 5z^2$
$\quad s^2 - 5t^2 = x^2 + 3xy + 2y^2$

Claim: if we choose any point $p \in X(L)$, can write A in a Zariski nbhd of p so that mult. constants are regular in nbhd & so can specialize A to p $\rightsquigarrow$ get a Quat. algebra as above.

i.e. can represent A near p as (f, g) where f, g are invertible, regular near p.

$$A = \left(5, \frac{(x+y)}{x}\right) \quad \text{over } k(X)$$

$$\begin{aligned} &\quad \wr\!\| \\ &\left(5, \frac{(x+2y)}{y}\right) \\ &\quad \wr\!\| \\ &\left(5, \frac{(x+y)}{y}\right) \\ &\quad \wr\!\| \\ &\left(5, \frac{(x+2y)}{x}\right) \end{aligned}$$

$$X: \begin{array}{l} s^2 = xy + 5z^2 \\ s^2 - 5t^2 = x^2 + 3xy + 2y^2 \end{array}$$

Suppose each presentation on left is bad at a divisor D

Theorem (Auslander): the "bad pts" as above are only in codim 1.

Useful facts

$(a,b)\otimes(a,c) \cong M_2((a,bc))$

Sketch: $(a,b)\otimes(a,c) \hookrightarrow F(\sqrt{a})\otimes F(\sqrt{a})$

$\underset{A}{\parallel}$ $\qquad B \qquad$ $\cong F(\sqrt{a})\times F(\sqrt{a})$

$\widehat{A} \cong \mathrm{End}_{\text{right } A\text{-mod}}(A) = \begin{bmatrix} \mathrm{Hom}(e_1A, e_1A) & \mathrm{Hom}(e_1A, e_2A) \\ & \end{bmatrix}$ $e_1 \qquad e_2$

$A = e_1A \oplus e_2A$

$\quad\; \underset{B}{\parallel} \quad \underset{B}{\parallel} = (a,bc)$

Cancellation (up to iso) $\quad Q_1\otimes Q_2 \cong Q_1\otimes Q_3$

$Q_1\otimes Q_1 \cong \mathrm{End}_{F}(Q_1) \quad \Longleftrightarrow Q_2 \cong Q_3$
vs.

$(x\otimes y)(z) = xz\bar{y}$ ← quat conj.

follows that $\cong$ classes of algebras which are $\otimes$ products of quat's is a commutative cancellative monoid.

Matrix algebras are a submonoid.
quat. is called $Br_2(F)$

$H^1(\mathrm{Gal}(F^{sep}/F), \mu_2(F^{sep}))$
$\shortparallel$
$H^1(F, \mu_2)$ (a)
$\shortparallel$
$\mathbb{Z}/2 = \mu_2^{\otimes 2}$

$Br_2(F)$
$\shortparallel$
$H^2(F, \mu_2)$

$H^*(F, \mu_2^*)$

$\mathbb{Q}$ $X(\mathbb{Q}) \neq \emptyset$

$$X(\mathbb{A}_\mathbb{Q}) = \prod_p X(\mathbb{Q}_p) \qquad X \text{ proj (smooth)}$$

$$X(\mathbb{Q}) \overset{\subset \cdots}{\subset} X(\mathbb{A}_\mathbb{Q})^{Br} \underset{\neq}{\subset} X(\mathbb{A}_\mathbb{Q})$$
$= \emptyset$ is $\neq \emptyset$

Ramification / tame symbol $\quad$ F discretely valued field w/ valuation v $(2 \neq 0)$
res field k

$$H^1(F, \mu_2) \longrightarrow H^0(k, \mu_2) = \pm 1$$
$$F^*/(F^*)^2 \quad (a) \longmapsto v(a) \bmod 2$$

$\mathbb{Q}_p$ $v = p$-adic
$k = \mathbb{F}_p$

$$Br_2(F) = H^2(F, \mu_2) \xrightarrow{\sim \text{ for } F = \mathbb{Q}_p} H^1(k, \mu_2) = k^*/(k^*)^2 = \pm 1 \text{ if } k = \mathbb{F}_p$$
$$(p \neq 2) \qquad (a, b) \longmapsto (-1)^{v(a)v(b)} \left(\frac{b^{v(a)}}{a^{v(b)}}\right)$$

$$H^2(\mathbb{Q}_p, \mu_2) = \mathbb{Z}/2\mathbb{Z}$$

ex: $\mathbb{Q}_p$ let $\bar{a} \in \mathbb{F}_p^*$ a nonsquare $a \in \mathbb{Q}_p$ a lift.

$$(a,p) \longmapsto (-1)^{v(a)v(p)} \overline{\left(\frac{p^{v(a)}}{a^{v(p)}}\right)} \qquad \begin{matrix} v(a)=0 \\ v(p)=1 \end{matrix}$$

on $\mathbb{R}$ $(-1,-1)$ $\qquad \qquad = \overline{a^{-1}}$

Brg theorem:

Albert-Brauer-Hasse-Noether theorem.

$$0 \to Br_2(\mathbb{Q}) \to \bigoplus_p Br_2(\mathbb{Q}_p) \xrightarrow{\Sigma} \pm 1 \to 0$$

$(a, p^2) \to 0$ in tame symbol $\Rightarrow 0$ in $Br_2 \Rightarrow$ $M_2(\ )$

If $Q \in X(\mathbb{A}_\mathbb{Q})$ $\quad Q = (Q_p)$ $\quad$ powers vals of Q.

given quat alg A on X (on $\mathbb{Q}(X)$ representable w/out poles / zeros)

can consider A_{Q_p} algebra over $\mathbb{Q}_p$

and compute if it is ± 1 in $Br_2(\mathbb{Q}_p)$

if these don't ~~add~~ multiply to 1 $\Rightarrow$ we say $Q \notin X(\mathbb{A}_\mathbb{Q})^A$

$$A = (5, (x+y)/x)$$
$$\cong$$
$$(5, (x+2y)/y)$$
$$\cong$$
$$(5, (x+y)/y)$$
$$\cong$$
$$(5, (x+2y)/x)$$

Step 1: suppose $\sqrt{5} \in \mathbb{Q}_p$.
then if $Q \in X(\mathbb{Q}_p)$
then A_Q is a trivial algebra.

$Q = [s_0; t_0; x_0; y_0; z_0]$ ✓

$\Rightarrow A_Q \to$ invariant $+1$

$$(a^2, b) \approx (a, b) \otimes (a, b) = M_4(\ )$$

Step 2: if p is inert in $\mathbb{Q}(\sqrt{5})/\mathbb{Q}$
if 5 not a square mod p $(\neq 5)$

$Q: (s_0; t_0; x_0; y_0; z_0)$ if $v(x_0) \geq v(y_0)$

$$0 = v(1) = v\left(\frac{x_0+2y_0}{y_0} - \frac{x_0+y_0}{y_0}\right) \geq \min\{\underbrace{v(x_0/y_0+2, \tfrac{x_0}{y_0}+1)}_{\geq 0}\}$$

since $v(x_0) \geq v(y_0)$

so either $v(x_0/y_0 + \underset{\text{or } 2}{1}) = 0 \Rightarrow$ tame symbol
A trivial $+1$

Final step: for $v=5$ must be nontrivial at every ($\mathbb{Q}_5$ pt)

$\Rightarrow$ mod 5 $s_0^2 \equiv x_0 y_0 \equiv x_0^2 + 3x_0 y_0 + 2y_0^2$

$\vdots$