Graph Theory, Spring 2016, Homework 2

1. Show that G is connected if and only if we cannot find nonempty subgraphs H_1, H_2 such that G is a disjoint union of H_1 and H_2.

2. Recall that $c(G)$ is the number of components of the graph G, and for a vertex $v \in V_G$, $G - v$ is the graph obtained by removing the vertex v and all its incident edges (more formall, $G - v = G[V_G \setminus \{v\}]$.

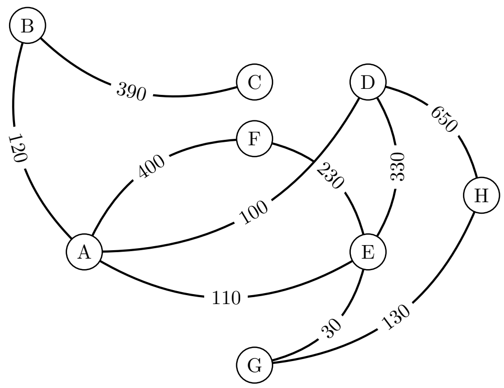
   Suppose G is a graph and $v \in V_G$ with $\deg(v) = 1$. Then show that $c(G) = c(G - v)$.

3. Recall that a bridge is an edge e in a graph G such that $c(G - e) > c(G)$. Show that if G has 7 vertices and is connected, then it must have at least 6 edges.

4. Show that if a simple graph G has 7 vertices, and at least 16 edges, then it must be connected, and if it has 7 vertices and at least 17 edges, then it cannot have any bridges.

5. (6000 level) More generally, show that if a simple graph G satisfies $e(G) > \binom{v(G)-1}{2}$ then it must be connected, and if $e(G) > \binom{v(G)-1}{2} + 1$ then it cannot have any bridges.

6. Consider the following graph, with weights assigned to the edges:

   Apply Dijkstra's algorithm to find a minimal path from the vertex H to the vertex B. Illustrate the sequence of graphs $T_0, T_1, \ldots$ which arise during the process.

7. (6000 level) Suppose that one uses Dijkstra's algorithm to, starting from a vertex v in a graph G, construct a tree $T < G$ containing a minimal length path from the vertex v to some other vertex $w \in V_G$.

   (a) If $v' \in V_T$, and if W is a (v', w) path in T which does not pass through v, is $\ell(W) = d(v', w)$? That is, is W a minimal path from v' to w?

   (b) As with the notation of the previous question, let U be the unique path in T from v to w and U' the unique path in T from v to v'. Let $H < G$ be the subgraph obtained as the union of the paths U and U'.

   Show that W in the previous part must be minimal if every vertex of H has degree at most 2 in H.