Lecture 11: connectivity, final part (a)

Tuesday, February 16, 2016 12:32 PM

Menger's theorem (Vertex version) G graph

Suppose, $u, v \in V_G$ nonadjacent. Then

$\kappa = \kappa(u,v) = p(u,v)$

Last time:

noticed that $\kappa(u,v) \geq p(u,v)$

still need to show, if k is min'l number of vertices needed to be removed to disconnect u from v,

proved by induction on $e(G)$ then we can find k disjoint (u,v)-paths.

First case: considered the possibility that every edge in G is incident either to u or to v.

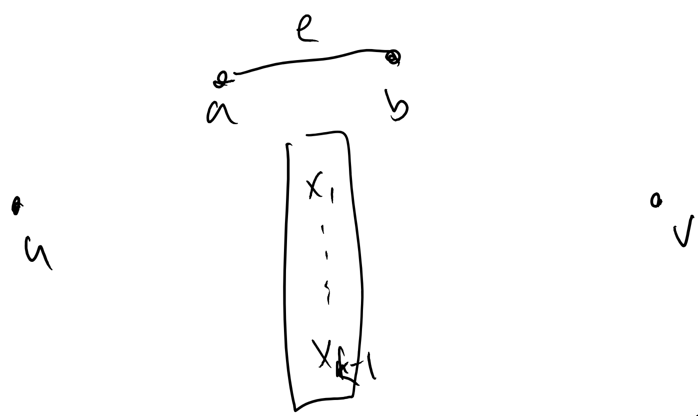
Remains to consider case where $\exists$ at least 1 edge not inc. to u or v.

Suppose $\exists$ e connecting a & $b \in V_G$ not incident to u or v

Showed that can assume in $G' = G - e$, can remove exactly $k-1$ vertices $x_1, \dots, x_{k-1}$ so that

$G' - \{x_1, \ldots, x_{k-1}\}$ has no $(u-v)$-paths

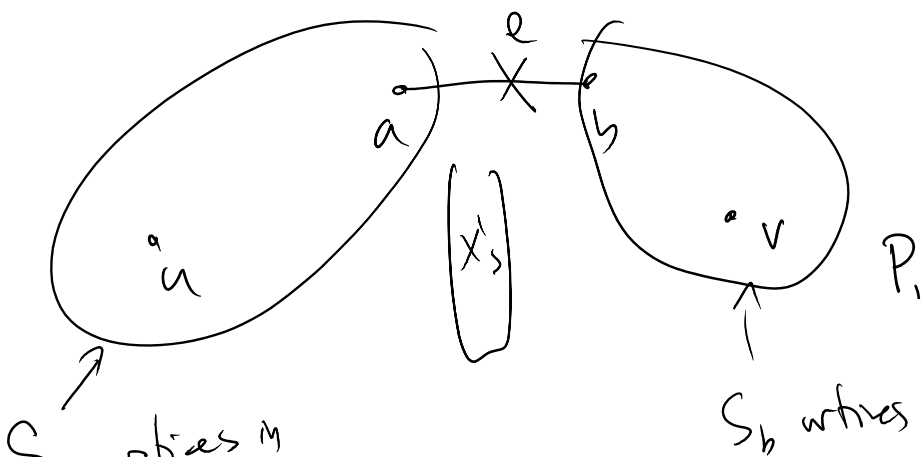
since $k = \kappa_G(u,v)$, we know that there <u>are</u> (u,v)-paths in $G - \{x_1, \ldots, x_{k-1}\}$



so only ~~way~~ paths from u to v are through e.

assume that path goes $u \leadsto a \leadsto b \leadsto v$.

after removing e,
u, a in same component
and v, b in same component
of $G - \{x_1, \ldots, x_n\} - e$.



in G', by induction have disjoint paths $P_1, \ldots, P_{k-1}$ from u to v.

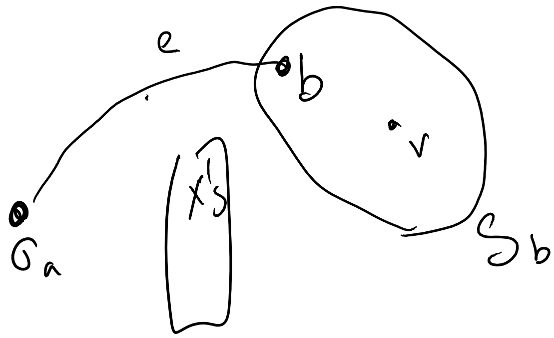
S_a vertices in this comp. of $G' - \{x_1, \dots, x_{k-1}\}$

S_b vertices in this comp.

Consider the graph G/S_a . notice, this has fewer edges, since u is in same comp as a S_a there is a sequence of edges connecting vertices in S_a which are lost.

new vertex σ_a

also, notice $\kappa_{G/S_a}(\sigma_a, v) = \kappa_G(u,v) = k$



why? well, $\{x_1, \dots, x_{k-1}, b\}$ is a (σ_a, v)-vertex cut

$\kappa_{G/S_a}(\sigma_a, v) \le k$

and if $\{y_1, \dots, y_\ell\}$ is a (σ_a, v)-vertex cut, then it is also a (u,v)-vertex cut in G

if $\vec{P}$ is a (u,v) walk in G missing the y's, then $P/S_a \cap P$ is a (σ_a, v)-walk in G/S_a.

so $k \le \kappa_{G/S_a}(\sigma_a, v)$.

By induction, can find k paths $P_1, \dots, P_k$ in G/S_a

vertex disjoint (σ_a, v)-paths

Since $G/S_a - \{x_1, \ldots, x_{k-1}\} - e$ has no (σ_a, v)-paths

we find all paths must either involve an x_i or e.

$P_1 = Q_1 Q_1'$ $\quad$ Q_1 (σ_a, x_1) path Q_1' (x_1, v) path

P_2 $\quad$ Q_i (σ_a, x_i)-path Q_i' (x_i, v) path

$\vdots$

$P_{k-1} = Q_{k-1} Q_{k-1}'$ $\quad$ Q

$P_k = \sigma_a e b Q_k'$ $\quad$ Q_k' a (b, v)-path

do same for S_b.

$R_1 = T_1 T_1'$ $\quad$ T_1 (σ_b, x_1) path T_1' (x_1, u) path

R_2 $\quad$ $\vdots$

$\vdots$

R_{k-1}

$R_k = \sigma_b e a T_k'$ $\quad$ T_k' is a (a, u)-path.

to finish, we use k paths:

$(T_1')^{-1} Q_1'$

$(T_2')^{-1} Q_2'$

$\vdots$

$$\vdots$$

$$(T'_{k-1})^{-1}\, Q'_{k-1}$$

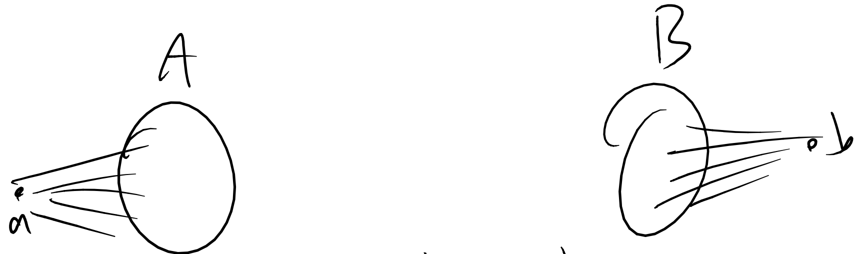
$$(T'_k)^{-1}\,(aeb)\,(Q'_k)$$

Generalization

Prop $A, B \subset V_G$ disjoint w/ size at least $\kappa(G)$, G connected

then $\exists$ $\kappa(G)$ internally disjoint paths from A to B.

Pf:



add new vertices a, b as above

a adj to everything in A,

b — — B.

$\square$