Lecture 12: connectivity, final part (b). Eulerian circuits

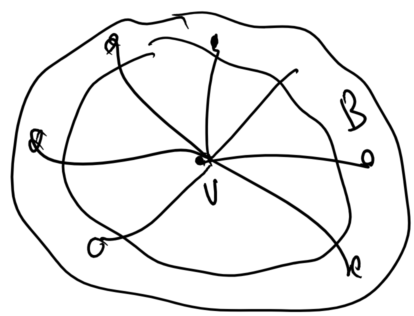
Thursday, February 18, 2016 12:42 PM

Thm (Dirac)

If G is k-connected ($k \geq 1$), then there exists a cycle passing through any given k vertices.

Def G is k-connected means $\kappa(G) \geq k$

Def a k-fan in a graph G from a vertex v to a set $B \subset V_G$ is a family of k internally disjoint paths from v to distinct vertices in B



Prop (Fan Lemma) If G is k-connected, $v \in V_G$ $B \subset V_G \setminus \{x\}$ w/ at least k vertices, then $\exists$ a k-fan from to v to B.

Pf: similar to argument from end of last lecture.

Details: HW.

Pf. of Dirac:

Suppose G is k-connected, given $v_1, \dots, v_k \in V_G$

want a cycle through these vertices.

Induct on k.

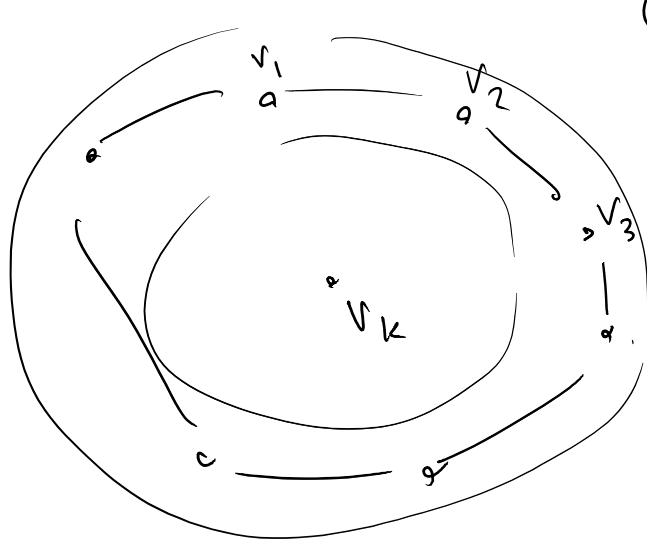
$k = 2$ ✓

$k \geq 3$ $\quad S = \{v_1, \dots, v_k\}$, $T = S \setminus \{v_k\}$

by induction can find a cycle through $v_1, \dots, v_{k-1}$

C

say $C = v_1 v_2 \cdots v_{k-1} v_1$

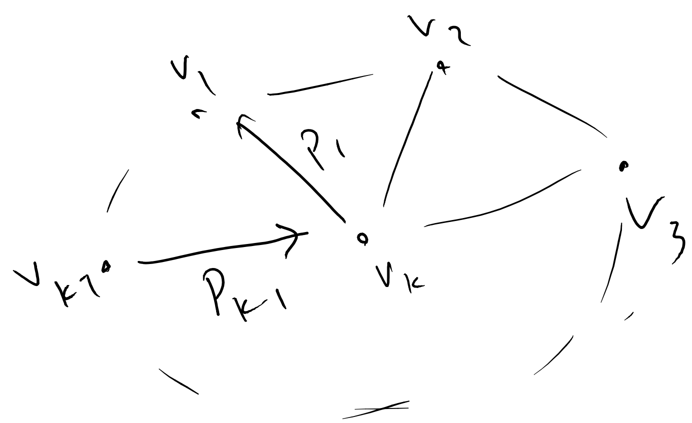


Case 1 : $V_C = \{v_1, \dots, v_{k-1}\}$

In this case, since G is $k-1$ connected, can find a $k-1$ fan from v_k to $V_C = \{v_1, \dots, v_{k-1}\}$

which gives in particular,

(v_k, v_1)-path P_1 and (v_{k-1}, v_k)-path P_{k-1}



new cycle

$v_1 v_2 \cdots v_{k-1} P_{k-1} P_1$

<u>Case 2</u> $\#V_C \geq k$, can find a k fan from v_k to C

suppose $C = P_{12} P_{23} P_{34} \cdots P_{k-2,k-1} P_{k-1,1}$

$P_{i,j}$ is a (v_i, v_j)-path

in total, have $k-1$ segments $P_{i,j}$

and fan has paths $Q_1, \ldots, Q_k$

from v to V_C, by Pigeonhole principle,

$\exists$ i,j s.t. Q_i & Q_j (and in same segment $P_?$

suppose Q_1 & Q_2 end in $P_{k-1,1}$

$P_{k-1,1} = v_{k-1} w_1 w_2 \dots, w_r v_1$ $\qquad v_{k-1} = w_0$

$v_1 = w_{r+1}$

Q_1 is a (v_{k}, w_i) path &

Q_2 is a $(v_k, w_{i+\ell})$ path

new cycle: $Q_2 w_{i+\ell} w_{i+\ell+1} \dots w_{r+1} P_{12} P_{23} \dots P_{k-2,k-1}$

$w_{r+1} = v_1$

$\dots v_{k-1} w_1 w_2 \dots w_i Q_1^{-1}$ $\qquad \Box$.

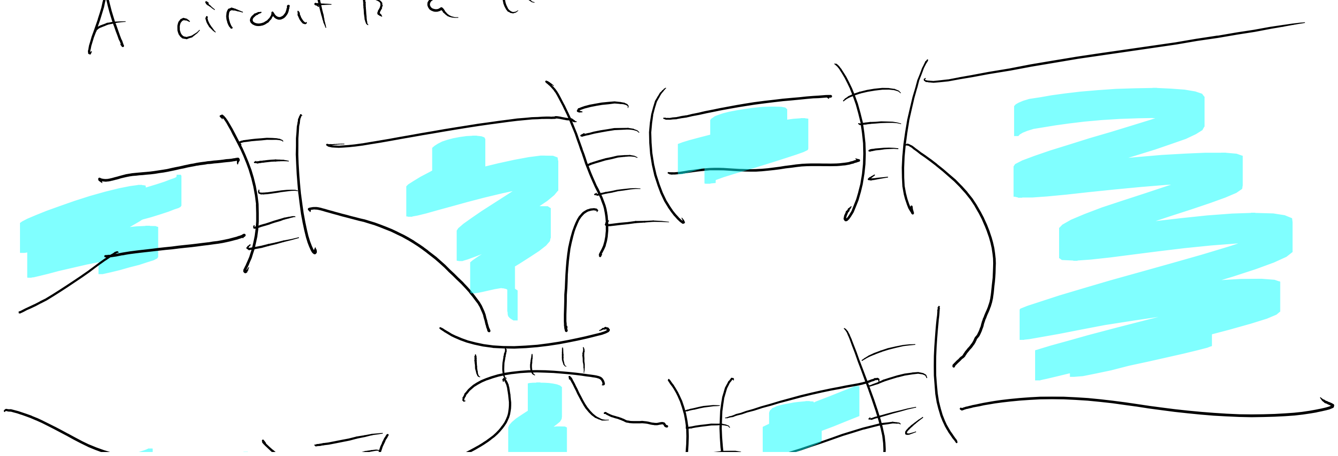
$v_{k-1} = w_0$

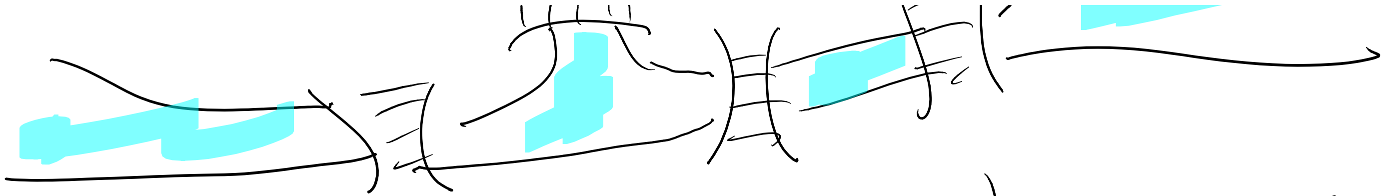
Let's turn to cycles & their relatives.

Recall A walk is closed if its origin and terminus coincide.

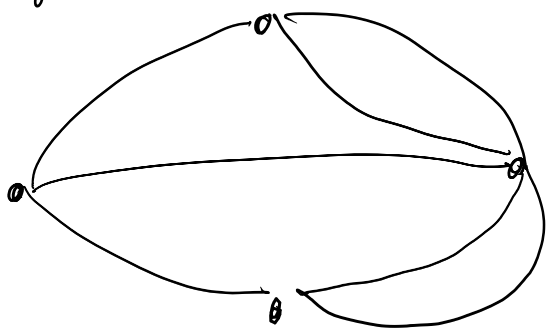
A trail is a walk w/ no repeated edges.

A circuit is a closed trail.





Can you start & end at same place, crossing each bridge exactly once? Königsburg bridge problem.



Euler's observation: if this was possible (for same graph) would need degree of each vertex to be even.

Def An Eulerian trail = trail which includes every edge.

An Eulerian circuit = closed Eulerian trail.

Thm A connected graph has an Eulerian circuit $\iff$ every vertex is even degree.

Pf: if $\exists$ odd degree vertex, we've explained already

— why it's impossible.