Lecture 13: Eulerian circuits, Hamiltonian cycles

Tuesday, February 23, 2016 12:37 PM

Last time:

Eulerian circuit (circuit = tour)
"
circuit using every edge in graph.

" closed trail
" walk w/ no repeated edges

Theorem: Let G be a nontrivial connected graph. Then G has an Eulerian circuit $\iff$ every vertex has even degree

- Let G be a simple graph. Then G has cycle subgraphs $H_1, \dots, H_n$ w/ G an edge disj. union of the H_i's $\iff$ every vertex in G has even degree.

Cor If G is simple, connected then TFAE

- Every vertex of G even degree
- G has an Eulerian circuit
- G is an edge-disjoint union of cycles.

Pf: because every vertex in a cycle has even degree in cycle $\Rightarrow$ G a disj. union of cycles implies all vertices even degree.

Conversely, G is simple, not a tree, connected $\Rightarrow$ G has a cycle $C \subset G$. Consider each comp. of $G - E_C$. By induction (on # of edges), each comp. is a union of cycles D.

For first part, if $\exists$ Eulerian circuit, then each vertex has even degree (= 2 · # times vertex arises in the circuit)

Conversely let C be a circuit of maximal length in G, consider $G - E_C$. if H is any nontrivial comp. in this then degree of every vertex in H is even.
by induction H has an eulerian circuit. can connect them. contradiction if H is nontrivial.

HW Problem: Let G be a connected graph, $W \subset V_G$ a subset of vertices. Then $G - [W, W]$ is the union of components of each $w \in W$.

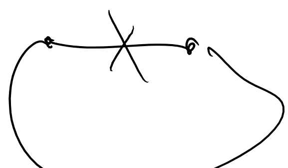
... includes every edge.

Def An Eulerian trail = trail includes every edge.

Prop G has an Eul. trail $\iff$ it has at most 2 odd degree vertices.

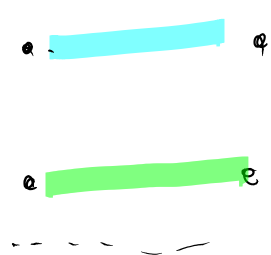
Pf: If G has 2 odd degree vertices, u, v. consider $G + uv \Rightarrow \exists$ eulerian circuit in $G+uv$.

remove $uv \rightsquigarrow$ get a trail.

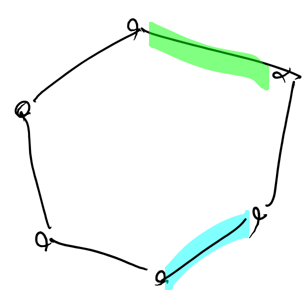


Ex: Suppose G connected has 4 odd degree vertices. show G is an edge disjoint union of two trails.

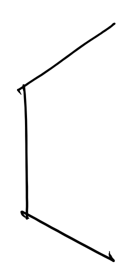
Pf



let $a, b, c, d \in V_G$ be odd vertices.
Set $G' = G + ab + cd$



let $C = v_1 e_2 v_2 \dots$ be an Eul-circuit in G'.



$\square$

$e_i = ab \quad e_j = cd$
same $i \neq j$
$i < j$ simplicity.

Hamiltonian Cycles

Def A Hamiltonian cycle in a graph G is a spanning cycle.

We say G is Hamiltonian if G has a Hamiltonian cycle.

NP: can check if someone exhibits a potential Ham. cycle. if they are right in poly time.
only exp-time algorithms are known to find one.

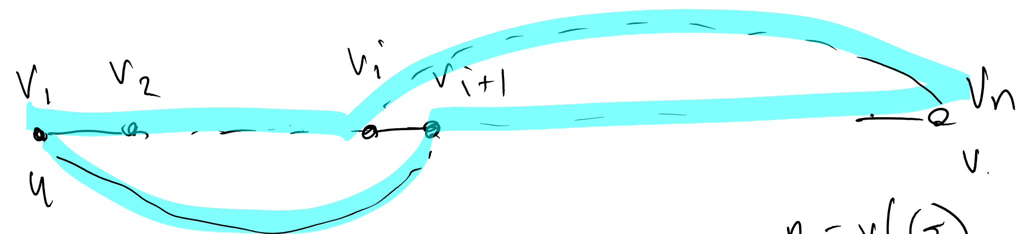
For some graphs its possible to get lucky.

Lem G simple, $u, v \in V_G$ nonadjacent, $d_G(u) + d_G(v) \geq v(G)$
Then G is Hamiltonian $\iff$ $G + uv$ is Hamiltonian.

Pf: G ham $\Rightarrow$ $G + uv$ ham.

$\Leftarrow$ $G + uv$ is Hamiltonian. either G was already ...

Suppose G+uv ...
Ham or can find a Ham cycle in G+uv using the edge uv. Remove uv, get a path from u to v using every vertex (Hamiltonian Path)



$n = v(G)$

$S = \{v_i \mid v_1 \text{ adj to } v_{i+1}\}$
$i<n$

$T = \{v_i \mid v_i \text{ adj to } v_n\}$

$v_n \notin S \cup T$

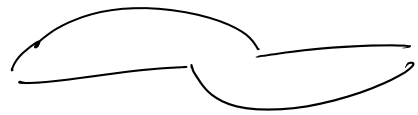
$\# S \cup T < n$

$\# S = \deg v_1 \quad \# T = \deg v_n$

$\# S + \# T = \deg \underset{u}{\underset{\shortparallel}{v_1}} + \deg \underset{v}{\underset{\shortparallel}{v_n}} \geq n = v(G)$

if $S \cap T = \emptyset$ then $\underset{n-1}{\underset{\not\geq}{\#(S \cup T)}} = \# S + \# T \geq n$
contradiction

$\Rightarrow S \cap T \neq \emptyset \Rightarrow \exists\, i$ s.t. v_{i+1} adj v_1 & v_i adj to v_n.

 D.

Algorithm: Given G, define $c(G)$ to be graph obtained by connecting all u,v s.t. $\deg u + \deg v \geq v(G)$
repeat. if we get K_n done!

ex:

Theorem (Dirac) If $\delta \geq v(G)/2$ then G is hamiltonian

Pf: $c(G) = K_{v(G)}$