Lecture 14: Vertex Colorings

Thursday, March 3, 2016 12:33 PM

G a graph a (vertex) coloring of G is a map

$V_G \to$ Some set.

an n-coloring is a (vertex) coloring whose range is $\{1, \ldots, n\}$

$f : V_G \longrightarrow \{1, \ldots, n\}$

An edge coloring is a map whose domain is E_G

n-edge coloring is $f : E_G \longrightarrow \{1, \ldots, n\}$

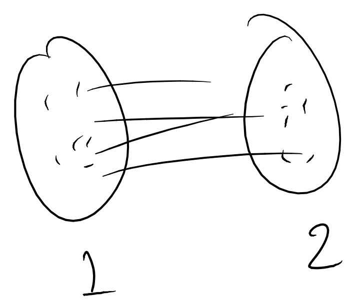
A vertex coloring $f : V_G \longrightarrow S$ is proper if $f(v) \neq f(w)$ whenever v & w are adjacent.

If G has a proper n-coloring, we say G is n-colorable.

<u>Def</u> $\chi(G) = \min\{n \mid G \text{ is } n\text{-colorable}\}$

<u>Ex</u>: 1-colorable = trivial

2-colorable = bipartite



<u>Def</u>: G is bipartite, if we can write $V_G = V_1 \cup V_2$ disjoint union.

s.t. $E_G = [V_1, V_2]$

$f: V_G \longrightarrow \{1,2\}$ proper $\Rightarrow$ define $V_i = f^{-1}(i)$

Def G is k-partite if can write $V_G = \bigcup_{i=1}^{k} V_i$ disjoint

s.t. $[V_i, V_i] = \emptyset$ all i.

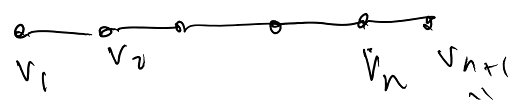
Notice: k-partite $\Leftrightarrow$ k-colorable.

Prop G is bipartite $\Leftrightarrow$ G has no odd length cycles. or loops.

Pf: G bipartite $\Rightarrow$ no loops ✓

if C odd length cycle in G

color of $v_1 = v_i$ i odd



v_n " same color as v_1 ; v_{n+1} " v_1 $\Rightarrow$ problem ✓

if G has no odd length cycles,

WLOG G connected Case 1: every vertex has even degree

• choose C an Eulerian circuit.

"

$v_1 v_2 \cdots v_n v_{n+1}$
$\quad\quad\quad\quad\quad\quad\quad\; \shortparallel$
$\quad\quad\quad\quad\quad\quad\quad\; v_1$

color v_i 2 if i even
1 if i odd.

If this definition of coloring makes sense, we are done! every edge arises, colors on each side are assigned diff colors (if this made sense)

Ex If every cycle in G has even length, an if G has no loops, then any circuit also has even length

(Induct on length) let C be minimal odd length circuit. If C a cycle, done. else, write $C = C_1 C_2 C_3$ where $C_2 = v_1 e_2 v_2 e_3 \cdots e_m v_m$ $\quad v_m = v_1$ first repeated vertex in C

C_2 cycle. (or bigon or something.)

Exercise $\Rightarrow$ assignment of colors as we traverse circuit is consistent.

~~Case 2~~:

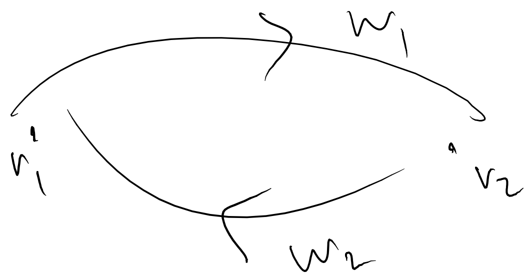
Induct on # odd degree vertices.

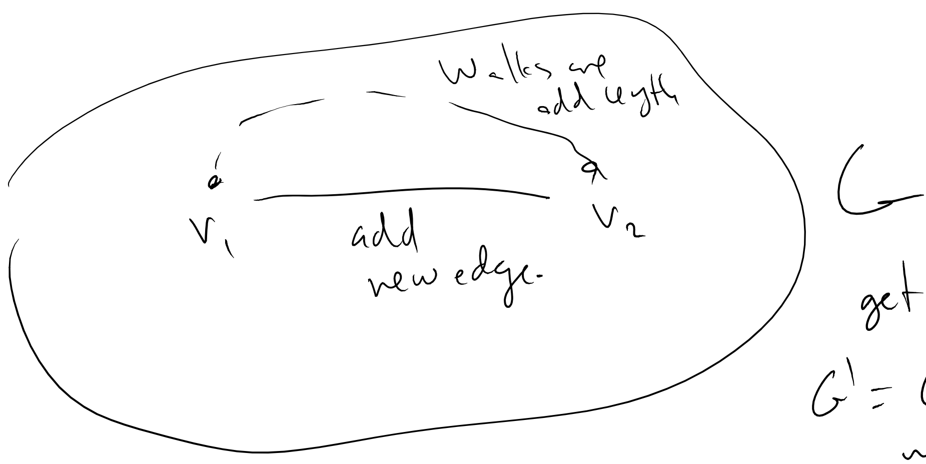
0 ✓

$2n$ odd deg vertices: v_1, v_2 be two of them

note: all walks from v_1 to v_2 have either even length or all have odd length.

Ex: Suppose G has no loops or odd length cycles. $v_1, v_2 \in V_G$. Then either every (v_1,v_2)-walk is odd length or every (v_1,v_2)-walk is even length.

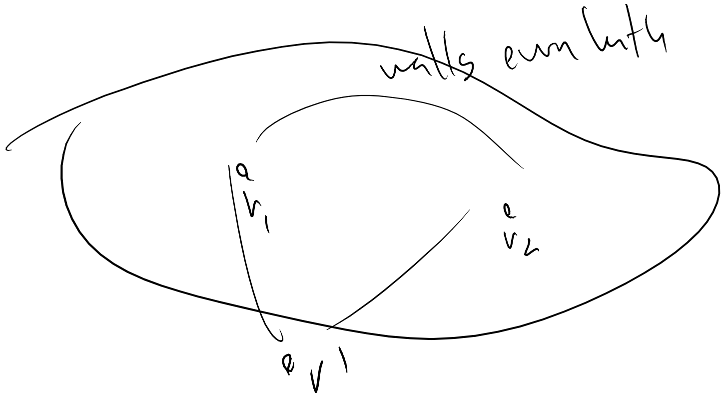




get new graph $G' = G + v_1v_2$ w/ same hyp. (check!)

but $2n-2$ vert. of odd dg.

but $2n-2$ vert. + any dg.



$G' = G^{+v'} + v_1v' + v_2v'$

by induction, G' bipartite

$f: V_{G'} \to \{1,2\}$

$f|_{V_G} : V_G \to \{1,2\}$

restrict coloring to G, so get G bipartite also. $\square$

<u>Def</u>: $\chi_G(x)$ = # of ways to color G properly w/ x colors.

<u>Facts</u>: $\chi_G(1) = \begin{cases} 0 & \text{if } G \text{ nontrivial} \\ 1 & \text{if } G \text{ trivial} \end{cases}$

$$\chi_G(2) = \begin{cases} 0 & \text{if } G \text{ is not bipartite} \\ 2^{c(G)} & \text{if } G \text{ is bipartite.} \end{cases}$$

<u>Ex</u>: $G = \bullet$ $\qquad \chi_\bullet(x) = x$

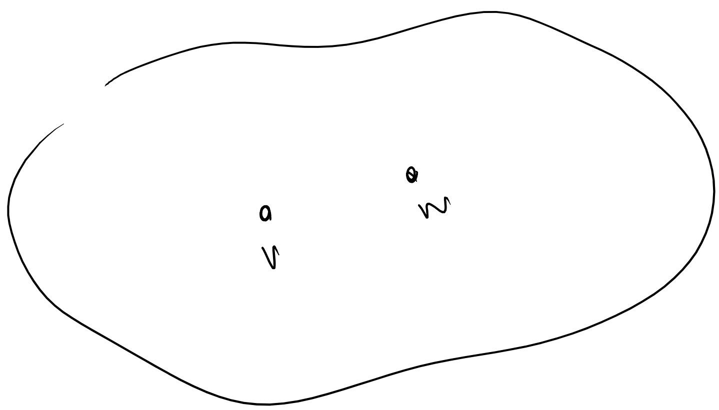
$\chi \quad (x) = x(x-1)$

χ  $(x) = x(x-1)$

χ  $(x) = x(x-1)(x-2)$

χ  $(x) = x(x-1)^2$

In fact, for any G, $\chi_G(x)$ is a polynomial.



if v, w are not adjacent

if $f: V_{G'} \to \{1, \dots, n\}$ is a coloring either

$f(v) = f(w)$ or $f(v) \neq f(w)$

f also colors $G'/\{v,w\}$

f also colors $G' + vw$

$$\chi_{G'}(x) = \chi_{G'/\{v,w\}}(x) + \chi_{G'+vw}(x)$$

Alternately, if $e \in E_G$ $\quad (G = G' + vw, \ vw = e)$

$$\chi_G(x) = \chi_{G-e}(x) - \chi_{G/e}(x)$$

$$\chi_{e}(x) = \chi(x) - \chi(x) = x^2 - x$$
$$= x(x-1)$$

$$\chi_{e}(x) = \chi(x) - \chi(x)$$
$$= \chi(x)\chi(x) - \chi(x)$$
$$= \chi(x)\big(\chi(x) - 1\big)$$
$$= x(x-1)(x-1)$$
$$= (x^2 - x)(x-1) = x^3 - 2x^2 + x$$