Lecture 15: Edge Colorings

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edge coloring:

$E_G \longrightarrow$ Set of colors

n-edge coloring: $E_G \longrightarrow \{1, \dots, n\}$

proper $\Leftrightarrow$ adjacent edges get different colors

$\chi'(G)$ = min # colors for a proper edge coloring.

observation: $\Delta(G)$ = max degree of a vtex

$\chi'(G) \geq \Delta(G)$ (when do we have $=$?)

(for loopless graphs)

Small Δ's:

$\Delta = 2 \Rightarrow$ each component is either

- trivial or ✓ 2 colors ok
- cycle or (yes if even length, no if odd!)
- path or ✓ 2 colors
- bigon or ✓ 2 colors
- ~~loop~~

$\chi'(G) = 2$ if no odd cycles

$= 3$ else.

$\Delta = 2$ or G is cubic graph (3-regular)

$\Delta = 3$ eg: G is cubic graph (3-regular)

yes if G has a Hamiltonian cycle.

$\chi' = 3$

(always loopless today)

Recall: G is a snark if it is a simple cubic graph w/ $\chi' = 4$

so, snarks don't have Hamiltonian cycles

Basic strategy for edge colorings: come up w/ colorings w/ "as many as possible" colors at each vertex.

proper coloring means # colors used at a vtx = degree of vtx.

Lemma G connected graph then either

- G has a 2 edge coloring s.t. both colors are represented at every vtx of deg at least 2. or
- G is an odd cycle.

Pf: If C is an Eulerian circuit in G then alternating colors works to give 2 colors to all internal vertices in circuit.

If all vertices have even degree, can apply this:

either
- G is a cycle ⟵ even cycle, ✓
- G has a vertex of deg ≥ 4.

↑

Let v have deg ≥ 4, choose Eulerian circuit starting & ending at v.

by degree consideration, v is also internal in this circuit, so strategy works.

If G has odd degree vertices, connect them & do above. ∎

If c is an edge coloring of G $\quad c: E_G \to \{1, \dots, n\}$

Def $c(v)$ = #colors used by edges incident to $v \in V_G$.

$\sum c(v)$ = "goodness of c"

$\sum_{v \in V} c(v)$ = goodness of c

proper $\Leftrightarrow$ $\sum c(v) = \sum d(v)$
$= 2e(G)$

Def c' an improvement of c is c' has more goodness than c