Lecture 16: Vizing's theorem

Thursday, March 17, 2016 12:37 PM

G a graph, no loops, $\Delta = \Delta(G)$ max degree of a vertex of G

Thm (Vizing) $\chi'(G) = \Delta$ or $\Delta+1$

Pf: Want to show $\chi'(G) \leq \Delta+1$ i.e. G is edge $\Delta+1$-colorable.

i.e. can find an edge coloring

$c: E_G \longrightarrow \{1, \dots, \Delta+1\}$ w/ $\Delta+1$ colors

s.t. it's proper: no adjacent edges get same color.

Go by induction on # of edges

Assume that every graph w/ fewer than n edges satisfies Vizing's thm ($\chi'(G) \leq \Delta(G)+1$)

Let G be a graph w/ n edges. want to show $\chi'(G) \leq \Delta(G)+1$

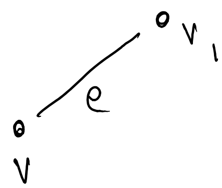
Strategy, choose $e \in E_G$, consider $G-e$.

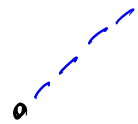
$\Delta(G-e) \leq \Delta(G)$ by induction, $G-e$ is edge $\Delta(G-e)+1$ colorable

$\Rightarrow$ $G-e$ is edge $\Delta(G)+1$ colorable.

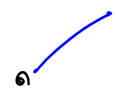
Can pick $c: E_{G-e} \to \{1, \dots, \Delta+1\}$ proper edge coloring.

visual notation:



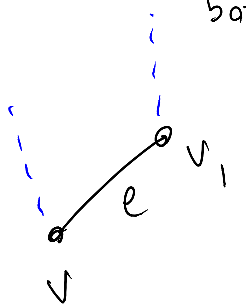


means, no blue edge at vertex.



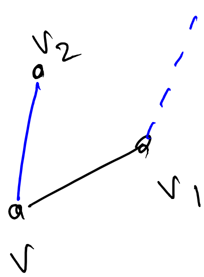
is a blue edge.

Case: $\exists$ color not appearing at both v_1 & v



in this case, can color e blue, done!

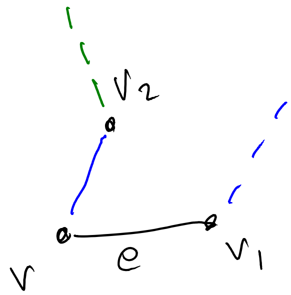
$\Delta+1$ possible colors to assign, each color arises either at v_1 or v.



$\exists$ color, which doesn't arise at v_1 and therefore, will have to arise at v.

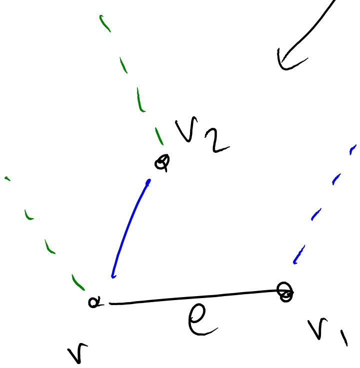
$\exists$ color which doesn't arise at v_2

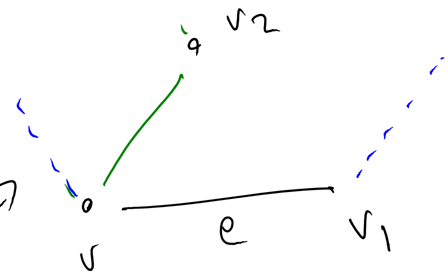
$\exists$ color which doesn't arise at v_2



if green doesn't arise at v

recolor blue edge vv_2 green

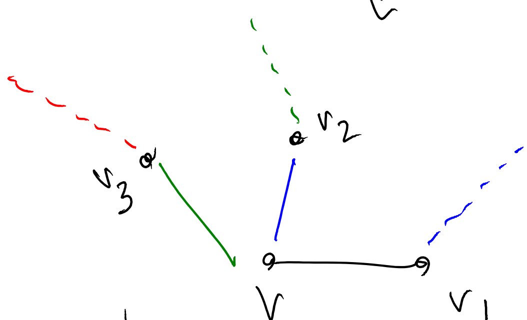




just recolored only blue edge.

so now, no blue at v.
$\Rightarrow$ can color e blue ✓

or, maybe v <u>has</u> a green edge!

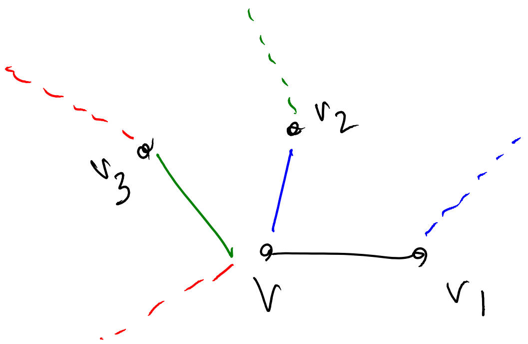


there is a color not at v_3

suppose it's a new color

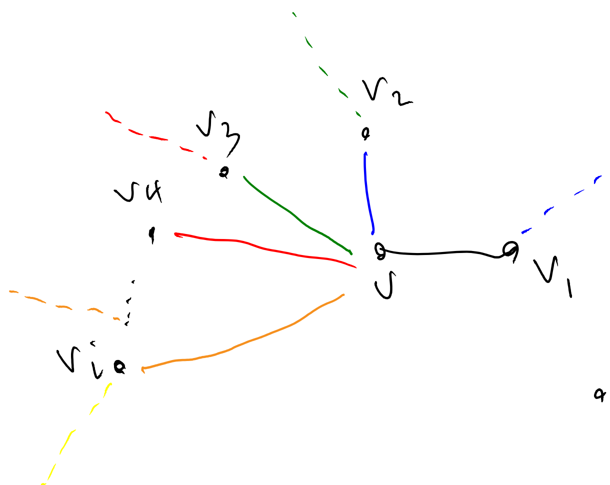
maybe v has no red edges...

or maybe it does have a red edge.



in this case, make vv_3 red, vv_2 green, vv_1 blue!

keep going.

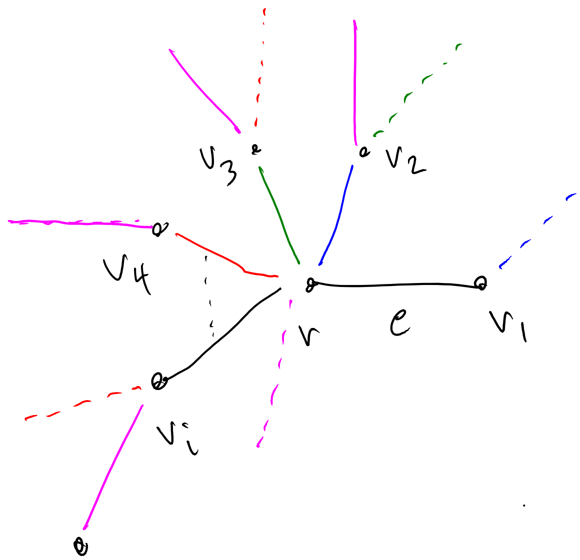


pattern continues until a run out of edges at v or a get a repeated color.

If yellow is a new color & out of edges at v, then yellow doesn't arise at v, so can recolor vv_i yellow, ... - recolor to eventually make vv_1

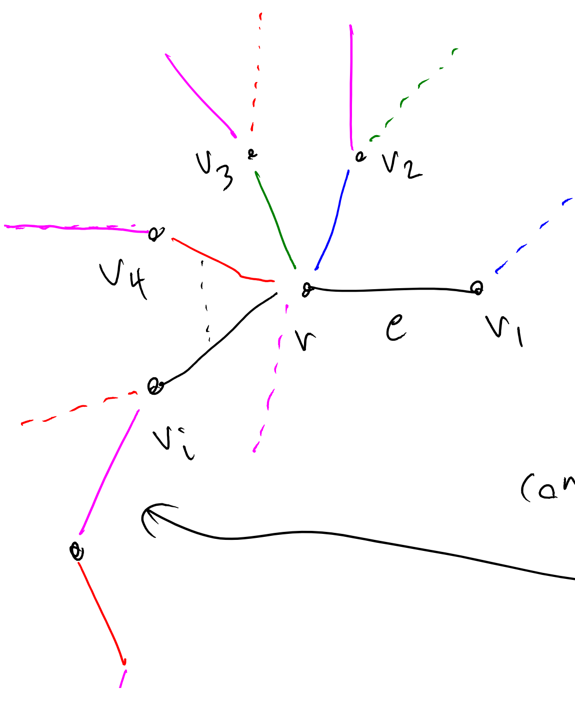
blue.

Can assume therefore that eventually, get a repeated color

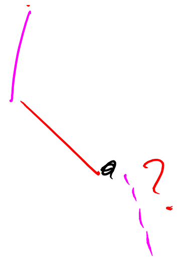


consider all red & purple edges. gives a subgraph where each vertex is deg 1 or 2 (or 0)

= paths, cycles, or isolated vertices.



consider part of subgraph containing this edge. (path)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