Lecture 17: edge coloring addendum, matchings

Tuesday, March 22, 2016 12:34 PM

By Vizing's theorem, now know

$\chi'(G) = \Delta(G)$ or $\Delta(G)+1$ (loopless)

G simple today.

Today we'll observe: $\chi'(G) = \Delta(G)$ if G is bipartite.

Recall G is bipartite $\iff$ G has no odd length cycles.

If $c: E_G \longrightarrow \{1, \dots, n\}$ edge coloring

recall $c(v) = \#$ colors for edges inc. to v.

$\sum_v c(v) =$ goodness

no more gooder $\iff$ optimal

Theorem If G is bipartite then $\chi' = \Delta$.

Pf: Suppose c is an optimal edge coloring w/ Δ colors.

suppose $c(v)$ has less than $d_G(v)$ colors.

there must be a repeated color "i" and an

unused color "j" at v.

Consider the edges $E_i \subset E_G$ colored i &
E_j colored j

let $H = G[E_i \cup E_j]$, consider the component
containing v.

The comp. containing v is not an odd cycle.

$\Rightarrow$ can 2-color edges of the comp so that
each color arises at least once at each vertex.

Do this w/ colors i, j. get c' new coloring. This is an improvement of
c ($c'(v) = c(v) + 1$, $c'(w) \geq c(w)$ all other $w \in V_G$)
contradicting optimality of c $\Rightarrow$ c is proper.

$\square$.

Matchings (aka 1-regular subgraphs)

Language:

Def A *matching* in a graph G is a subset
of edges $M \subset E_G$ s.t. no two elements of

non-loop of edges $M \subset E_G$ s.t. no two elements of M are adjacent.

<u>Def</u> If M is a matching on G, then we say a vertex $v \in V_G$ is <u>saturated</u> (by M) if v is incident to some edge in M

<u>Def</u> M is <u>perfect</u> if every vertex is saturated

<u>Def</u> M is a maximum matching if $|M| \geq |M'|$ for all matchings M'.

<u>Def</u> If M is a matching on G, a path is called M-alternating if edges alternate between "in M" & "not in M".

<u>Def</u> An M-alternating path is <u>M-augmenting</u> if its ends are unsaturated.

<u>Theorem</u> (Berge '57) M is a maximum matching ... paths.

Theorem (Berge '57) It is a ...

if and only if G has no M-augmenting paths.

Pf: If G has an M-augmenting path

v_1v_2 not in M, v_2v_3 in M, v_3v_4 not, ... $v_{2n-1}v_{2n}$ not in M

v_1, v_{2n} unsaturated then

define $M' = M \setminus \{v_2v_3, v_4v_5, \ldots, v_{2n-2}v_{2n-1}\} \cup \{v_1v_2, v_3v_4, \ldots, v_{2n-1}v_{2n}\}$

n-1 of these; n of these

is a match w/
$|M'| = |M| + 1$.

Conversely, suppose M is not maximum. want to show $\exists$ an M-augmenting path in G.

Typical situation: Focus on bipartite graphs

G w/ given bipartition $V_G = X \cup Y$

(no vertices in X are adj. to each other, sim for Y.)

$E_G = \{X, Y\}$ $\quad X \cap Y = \phi$.

<u>Q:</u> $\exists$ a matching saturating every vertex in X?

<u>Theorem</u> $\exists$ as above $\Longleftrightarrow$ $\forall$ subsets $S \subset X$

$|N(S)| \geq |S|$

$N(S) = \{$ vertices adj to vertices in $S \}$