Lecture 18: Matchings in bipartite graphs

Thursday, March 24, 2016 12:47 PM

Finding a matching in bipartite graph saturating one part "Hungarian Method"

<u>Procedure:</u> Start w/ G, $V_G = X \cup Y$ bipartition $E_G = [X, Y]$

want a matching saturates every $x \in X$.

Choose $u \in X$. let $H \subseteq G$ be graph w/ just u.

we'll grow H and a matching M (starting w/ $M = \emptyset$) to obtain a matching saturating more & more vertices of X.

At each stage H will have satisfy.

- All vertices of H are M-saturated except u or

given current H, M, let $S = V(H) \cap X$
$T = V(H) \cap Y$

note $N(S) \supseteq T$

if $N(S) = T$ then since each $t \in T$ is matched to a single $s \in S \setminus \{u\}$

$\Rightarrow |N(S)| \leq |S| - 1 = |S \setminus \{u\}|$

$\Rightarrow \nexists$ matching saturating $S \Rightarrow \nexists$ matching sat. X.
done: failed!

if $\exists\, y \in N(S) \setminus T$, say y is adj. to $x \in S$

Let's try again fixed G = bipartite $V_G = X \cup Y$.

— Set $M = \emptyset$

loop: if $\forall\, u \in X$ saturated, go home.

else given $u \in X$ unsat:

— Set H = graph w/ only u M-alternating.
(at each stage, H is tree)

$S = V(H) \cap X$, $T = V(H) \cap Y$

have $N(S) \supseteq T$

if $N(S) = T$ end, fail.

else $\exists\, y$ adj. to $x \in S$, $y \notin T$.

either: y is unsaturated

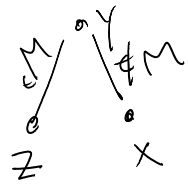
path from u to y is augmenting, use this to switch matching along path & sat. u

if y is saturated

M

if y is sat.

say $yz \in M$



add y & yz to H

H gets bigger

Cor of algorithm

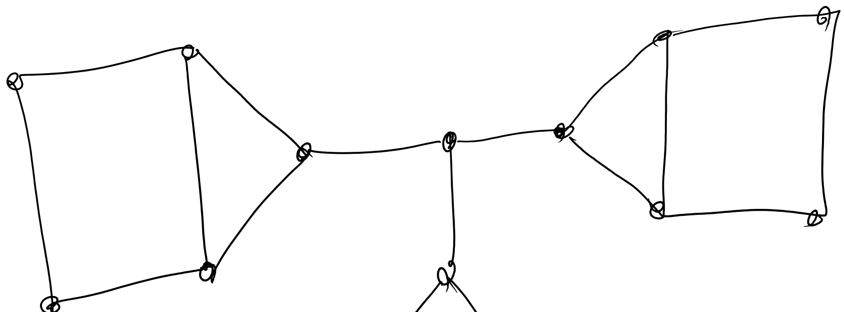
$\forall S \subset X, \ |N(S)| \geq |S| \iff \exists$ matching saturating X.

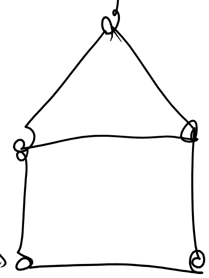
Cor of Cor: k-regular marriage problem.

If G is a k-regular bipartite graph $(k \geq 1)$ then $\exists$ perfect matching.

Pf: YouTube.

Matchings in non-bipartite graphs:



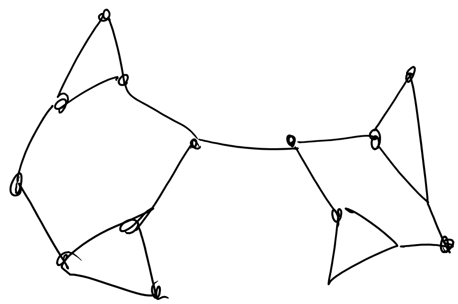


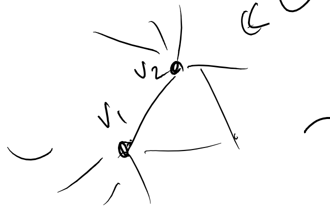
Def $o(G) = \#$ of comp. w/ odd # of vertices

Theorem (Tutte 1947)

G has a perfect matching $\iff \forall\, S \subset V(G)$

$o(G-S) \le |S|$





$G - \{v_1, v_2\}$