Lecture 19: flows and cuts

Tuesday, March 29, 2016 12:35 PM

The Plan: today - Marriage (briefly)
- Digraphs, flows, cuts (next tues ⇒ Menger stuff)

thurs - no class (video on alg. for matchings)

tues (April 5?) - HW due (not on 1st, thurs)

thurs (Ap 7?) - more HW due (to be assigned in two days)

Marriage problem

Thm If G is a bipartite graph which is k-regular then $\exists$ perfect matching.

Will use: Thm If G is bipartite w/ bipartition (X, Y) then $\exists$ matching saturating $X \iff \forall S \subset X$, $|S| \leq |N(S)|$

$N(S) = \{v \mid v \text{ adj to } s, \text{ some } s \in S\}$

Pf of thm:

Observe: if $S \subset X$, then if we let $\mathcal{E}_S = \{e \in E_G \mid e \text{ inc. to some } s \in S\}$

$|\mathcal{E}_S| = k|S|$

$$k|S| = \mathcal{E}_S \subset \mathcal{E}_{N(S)} = k|N(S)|$$

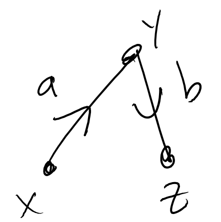
$$|N(S)| \geq |S| \quad \boxtimes.$$

<u>Note:</u> if a matchy M saturates $X \subset V_G$, (X, Y) bipartition $\Rightarrow |X| \leq |Y|$ since

$$X \longrightarrow Y$$
$x \longmapsto$ the y matched to x via M

injective by def of matching $\Rightarrow |Y| \geq |X|$.

<u>Digraphs</u> $D = (V, A, s, t)$ $\qquad s, t : A \to V$



$V_D = V = \{x, y, z\}$

$A_D = A = \{a, b\}$

$s(a) = x$, $t(a) = y$

$s(b) = y$, $t(b) = z$

$\text{indeg}(x) = 0$ $\quad \text{outdeg}(x) = 1$

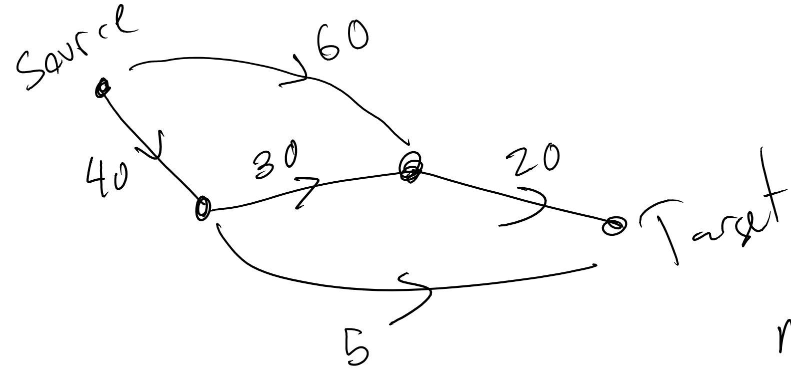
$\partial^-(x) = 0$ $\quad \partial^+(x) = 1$

given a digraph $D \rightsquigarrow$ get ordinary graph by forgetting directions of arrows.

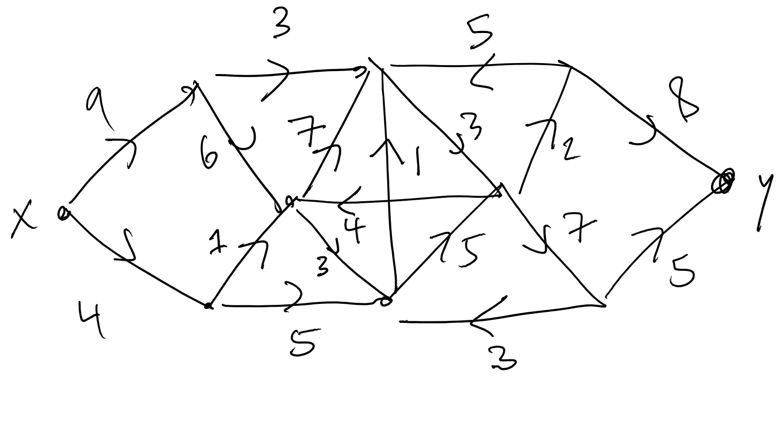
i.e. $E = A$ "the underlying graph of D"

A *walk* in D is a walk in the underlying graph of D

A *directed walk* in D is a sequence of vertices & arrows $x_1 a_2 x_2 a_3 \cdots a_n x_n$ where $s(a_i) = x_{i-1}$, $t(a_i) = x_i$



min-cut / max-flow theorem

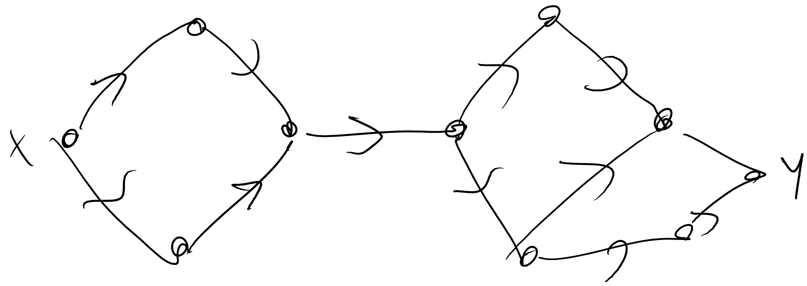


Notation:

Def A Network is a directed graph D together with a function $c: A \to \mathbb{R}_{\geq 0}$ and two distinguished vertices $x, y \in V_D$

will write $N = (D, c, x, y)$

c = "capacity function"

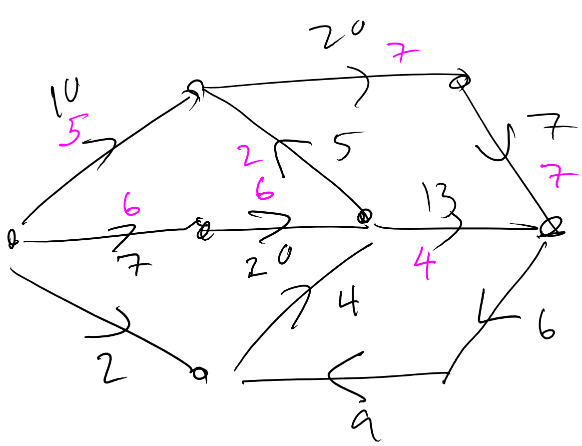


a flow (from x to y) is N is a function

$f : A \longrightarrow \mathbb{R}_{\geq 0}$ s.t. $f(a) \leq c(a)$ (feasability)

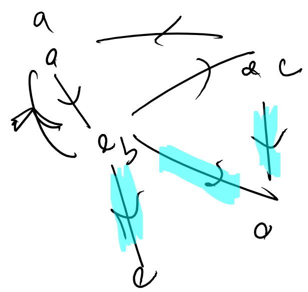
and s.t. $\forall\, v \in V$, $v \neq x, y$, we have

$$\sum_{\substack{a \text{ s.t.} \\ s(a) = v}} f(a) = \sum_{\substack{a \\ t(a) = v}} f(a) \quad \text{(conservation)}$$



Cuts: $[X, Y] = \{ a \in A \mid s(a) \in X, t(a) \in Y \}$

if $X \subset V_D$, set $\text{outcut}(X) = [X, V \setminus X]$



$X = \{a, b, c\}$

outcut(X)

$\text{incut}(X) = [V \setminus X, X]$

<u>Def</u> an (x,y)-cut in $N = (D, c, x, y)$ is cut of form outcut(X) where $x \in X$, $y \in N \setminus X$