Lecture 2: digraphs and degrees

Thursday, January 14, 2016 12:37 PM

$G = (V, E, \psi)$

simple graph – equivalent notion:

<u>Def</u> G is a graph. $G = (V, R)$

R = set of unordered distinct pairs of vertices

$(V,R) \rightsquigarrow (V, E, \psi)$

$E = R \quad \psi = \mathrm{id} : R \to R$

Formally explain why the above gives a graph from a graph

give a similar construction the other direction

simple graph $\rightleftarrows$ graph

<u>Alternately</u>

$G = (V, E, \alpha) \qquad \alpha \subseteq V \times E$

language: v incident to e if $(v,e) \in \alpha$.

graph $\Longleftrightarrow$ every edge incident to 1 or 2 vertices

$\uparrow$ simple if also every pair of vertices is incident to at most 1 edge.

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degree formula

Def A link is an edge which is not a loop

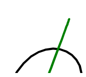
Def if G is a graph, $v \in V_G$, we define

$$\deg v = \deg_G v = d(v) = d_G(v)$$

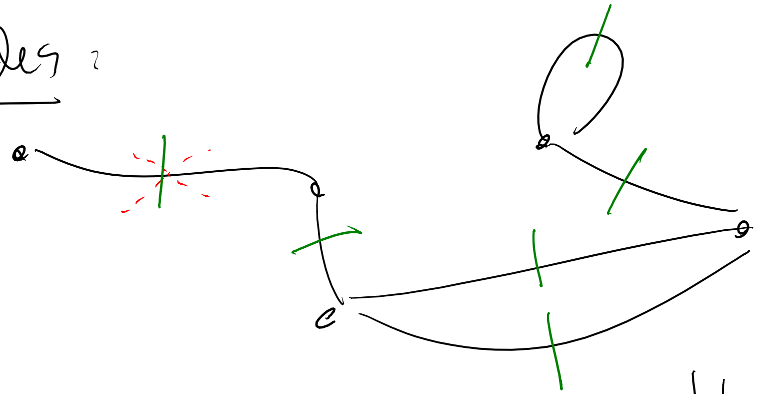
as $\deg v$ = # of links incident to v + 2 # of loops incident to v

Prop (degree formula)

$$\sum_{v \in V_G} \deg(v) = 2 \# E_G$$



Pf Idea:



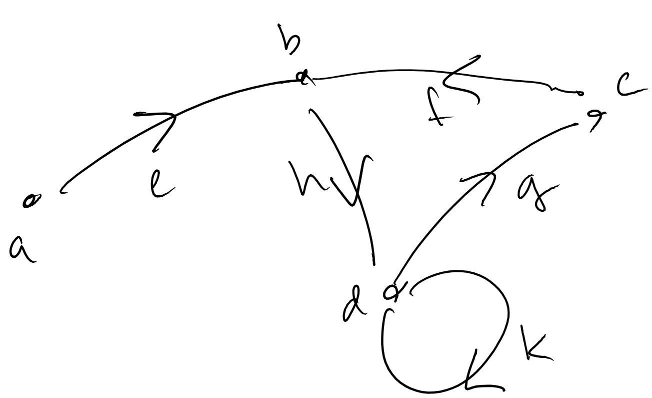
- each ½ edge contributes exactly 1 to the LHS
- 2 half edges for each edge.

Digraph = Directed Graph

Def A digraph is a quadruple $D = (V, A, s, t)$

V = set of vertices A = set of arrows

$s, t : A \longrightarrow V$ "source & target"



$s(e) = a$

$t(g) = c$

indegree$(v) = \#\{a \in A_D \mid t(a) = v\}$

$$\operatorname{outdg}(v) = \#\{a \in A_D \mid s(a) = v\}$$

<u>Prop</u> (degree formula for digraphs)

$$\sum_{v \in V_D} \operatorname{outdg}(v) = \sum_{v \in V_D} \operatorname{indg}(v) = \#A$$

<u>Pf:</u>

$$\sum_{v \in V_D} \operatorname{outdg}(v) = \sum_{v \in V_D} \#\{a \mid s(a) = v\}$$

$$= \sum_{v \in V_D} \sum_{a,\, s(a) = v} 1$$

$$= \sum_{(a,v),\, s(a) = v} 1$$

$$= \#\{(a,v) \mid s(a) = v\}$$

$$= \#A \qquad \square$$

<u>Proof of dg formula:</u>

given G, define D by $V_D = V_G$

$$A_D = \{ (e,v) \in E_G \times V_G \mid e \text{ a link}, v \text{ incident to } e \}$$
$$\cup \{ (l, yes), (l, no) \mid l \text{ loop} \}$$

$s(e,v) = e \qquad s(l,yes) = s(l,no) = t(l,yes) = t(l,no)$

$t(e,v) = w$ where $vw = \psi(e)$

Notice: D is a digraph, $\# A_D = 2 \# E_G$

$\text{outdg}_D(v) = \text{dg}_G(v)$

$$\sum_v \text{dg}(v) = \sum_v \text{outdg}(v) = \# A_D = 2 \# E_G \quad \square$$

Complete graph w/ n vertices

simple graph w/ n vertices s.t. every pair of vertices are connected by an edge.

Def a pair of vertices $v, w \in V_G$ are adjacent if they are connected by an edge.

Def if G_1, G_2 are graphs, an isomorphism

$f: G_1 \longrightarrow G_2$ is a pair of functions

$f_V: V_{G_1} \longrightarrow V_{G_2}$ $\quad f_E: E_{G_1} \longrightarrow E_{G_2}$

which are bijective and such that

$\underset{\in V_{G_1}}{v}$ incident to $\underset{\in E_{G_1}}{e}$ if and only if $f_V(v)$ incident to $f_E(e)$ in G_2

Lem/Observation

if G_1 & G_2 are simple graphs and $f: G_1 \longrightarrow G_2$ is an isomorphism then f is completely determined by f_V.