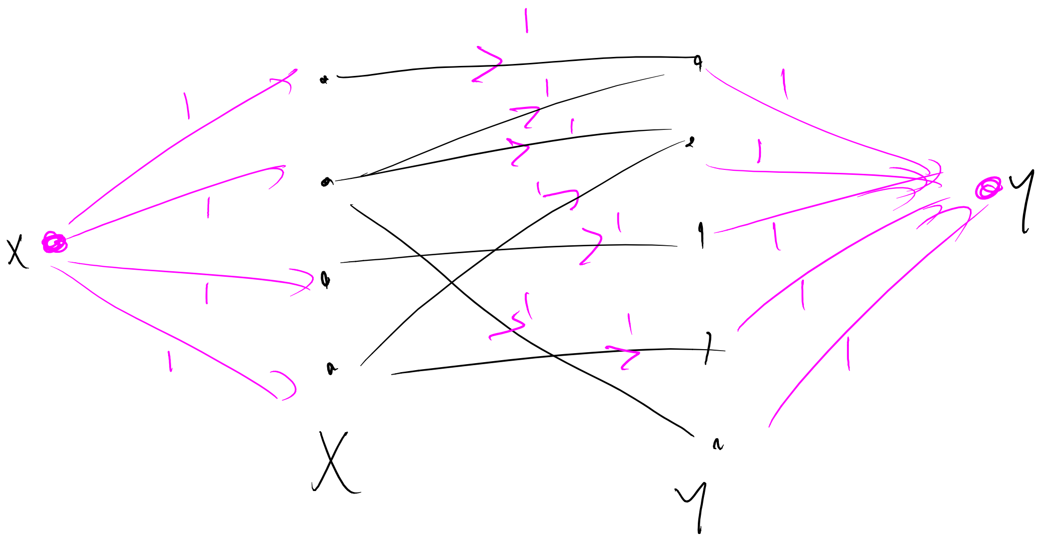
Lecture 20: min cut/max flow 1

Tuesday, April 5, 2016 12:48 PM

Algorithm for max-flow $\Longrightarrow$ algorithm for matchings in a bipartite graph



$\exists$ matching saturating X $\Longleftrightarrow$ $\forall S \subset X$, $|N(S)| \geq |S|$

[size of flow rep. by max matching = size min cut disconnecting x from y

min-cut max-flow says:

max flow = min # edges needed to cut to disconnect x from y (directed)

//

max size of a matching.

when is max size $= |X|$?

1-1 + 2 matchs

when is max size $= |X|$?

Q: can we prove using / that $\exists$ matching saturating X if $|S| \leq |N(S)|$ all S?

Given a network (D, c, x, y)

and a flow $f: A_D \to \mathbb{R}_{\geq 0}$

Given a set of vertices $X \subset V_D$

$$f^+(X) = \sum_{a \in \text{outcut}(X)} f(a)$$

$$f^-(X) = \sum_{a \in \text{incut}(X)} f(a)$$

net flow out of $X \equiv f^+(X) - f^-(X)$

$$\text{val}(f) = f^+(\{x\}) = f^-(\{y\})$$

<u>Prop</u> If f is any flow an $X \subset V_D$ w/ $x \in X$ $y \notin X$ then $v(f) =$ net flow out of X

Given any cut $K = \text{outcut}(X)$ $x \in X$ $y \notin X$

define $\text{cap}(K) = \sum_{a \in K} c(a)$

we want to show $\min\{\text{cap}\, K \mid K = \text{outcut}(X) \ x \in X \ y \notin X\}$

$$\shortparallel$$

$\max\{v(f) \mid f \text{ a flow}\}$

<u>Thm</u> $v(f) \leq \text{cap}\, K$ K as above.

<u>Pf</u>: Pick f $K = \text{outcut}(X)$

$v(f) = \text{netflow out of } X \leq \text{cap}\, K$

$\shortparallel$

$f^+(X) - f^-(X) \leq f^+(X) \leq \text{cap}\, K$ Δ.

notice $v(f) = \text{cap}\, K$ only if $f^-(X) = 0$
and $f^+(X) = \text{cap}\, K$

<u>Cor</u> If f a flow K a cut $v(f) = \text{cap}(K)$
then f is a max flow & K is a min cut.

Pf: let f^* be a max flow K^* min cut.

then $v(f) \leq v(f^*) \leq cap(K^*) \leq cap(K)$

$\Rightarrow v(f) = v(f^*) \Rightarrow f$ max

$\Rightarrow cap(K) = cap(K^*) \Rightarrow K$ min.

To finish: you can tell optimality of a flow by looking along x-y paths. (non directed)