Lecture 21: min cut/max flow 2

Thursday, April 7, 2016 12:34 PM

- No class tuesday April 12
- Exam #2 April 21

- Finish Min Cut/Max flow & Ford Folkerson algorithm
- Posets & trees
- Application to Menger type stuff.

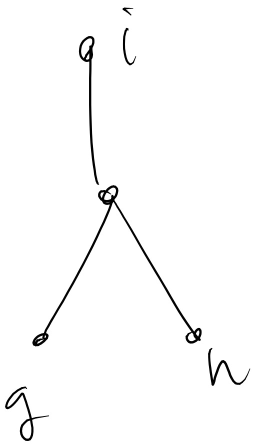
Def A poset (partially ordered set) is a set S together with a relation $\leq \subseteq S \times S$

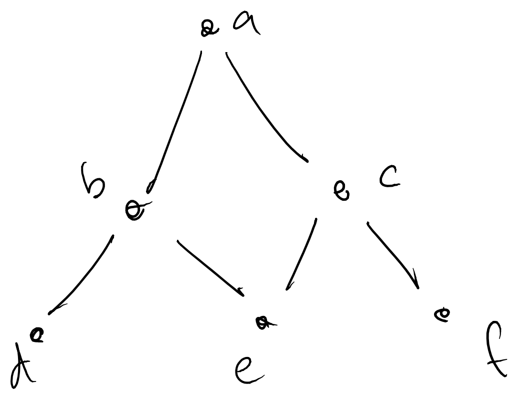
notation $a \leq b$ to mean $(a,b) \in \leq$

such that:

1) reflexive: $a \leq a$
2) transitivity: $a \leq b,\ b \leq c \Rightarrow a \leq c$
3) antisymmetric: $a \leq b,\ b \leq a \Rightarrow a = b$

Visually: diagram where higher is "bigger"





$a \geq b$

$e \leq c$

$e \leq a$

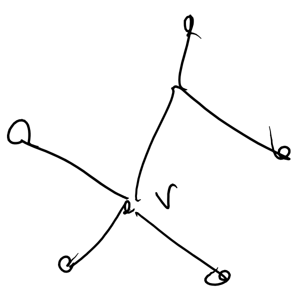
$e \nleq f$ & $f \nleq e$!

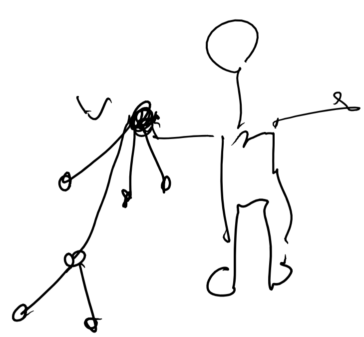
example $S = \mathbb{Z}_{>0}$

define poset structure via $|$ $\quad a|b \iff (a \leq b)$

example: if G is a tree, $v \in V_G$ can define a poset

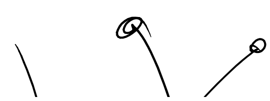
(visually hang tree by holding it up by v.)



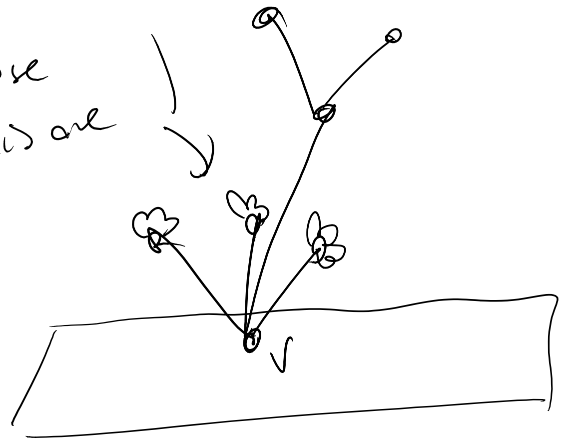


upside down

(we'll use



(we'll use this one)



"tree"

v "root"

Def A <u>rooted tree</u> is a tree together w/ a specified vertex. (G,v)

Given a rooted tree (G,v), define a poset via $S = V_G$, $a \leq b$ if $\exists$ a (v,b) path in G passing through a.

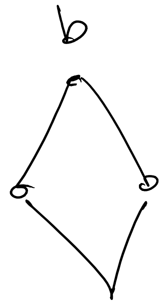
What says when a poset is a tree?

only <u>1 way to get to the root</u>.

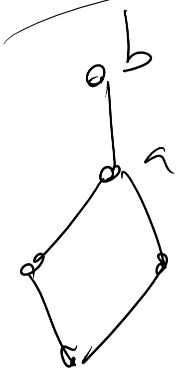
Def $a < b$ means $a \leq b$ & $a \neq b$.

Def If $(S,\leq)$ a poset, we say a is the ... lower ...

Def: ... predecessor of b if $a < b$ and whenever $c < b$ we have $c \leq a$.



b has no predecessor



a is b's predecessor

Exercise 1) trees are posets w/ predecessors. (except for min'l elmt) "root."

2) If A poset has predecessors (for all but min'l ones) then partial order is determined by predecessors!

Back to max flow / min cut

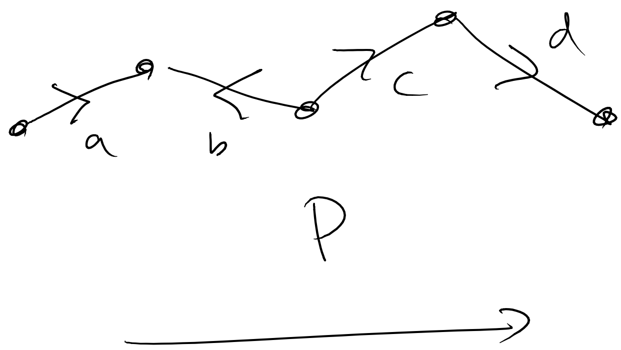
N = network = (D, c, x, y)

flow $f : A_D \to \mathbb{R}_{\geq 0}$ (conservative & feasible)

Given some N, f

if P is a path (undirected)

"forward arcs in P" "backwards arcs in P"



a, c, d forward arcs

b backward arc.

we say forward arc a in <u>unsaturated</u> if $f(a) < c(a)$

we say backward arc is <u>unsat.</u> if $f(a) > 0$

<u>Def</u> A path is unsaturated, if each of its arcs is unsaturated..

<u>Def</u> P is <u>increasing</u> if it is an unsaturated (x,y) path

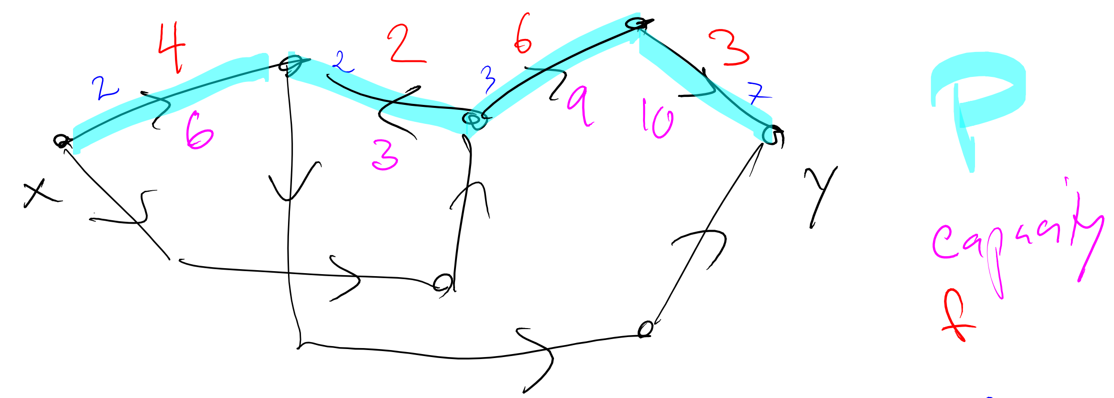
<u>Rem</u> If f admits an increasing path P

set $\varepsilon(P) = \min\left(\{ c(a) - f(a) \mid a \text{ forward} \} \cup \{ f(a) \mid a \text{ backwards} \} \right)$

then defining $f'(b) = \begin{cases} f(b) & \text{if } b \text{ not in } P \\ f(b) + \varepsilon(P) & \text{if } b \text{ forward in } P \end{cases}$

then a[...]g [...] $\begin{cases} f(b)+\varepsilon(P) & \text{if } b \text{ [...] in } P \\ f(b)-\varepsilon(P) & \text{if } b \text{ backward in } P. \end{cases}$

ex:



$\varepsilon = \min$
$= 2$

Prop If $\exists$ an f increasing path $\Rightarrow$ f is not maximum

Pf as above, replace f by f', $v(f') = v(f) + \varepsilon(P)$.

Prop If f is a flow w/ no f-increasing paths, if we set X = all vertices in D s.t. $\exists$ an f unsaturated path from x, then $\partial^+ X$ is a min cut & f is a max flow.

Recall $v(f) \leq cap(\partial^+ X)$ $\quad x \in X, y \notin X$.

$v(f) = f^+(X) - f^-(X) \leq f^+(X) \leq c^+(X) = cap(\partial^+ X)$

tot cap

tot cap
of outgoing
arcs from X

want to show
(by previous lecture)

$v(f) = cap(\partial^+ X)$

i.e. $f^-(X) = 0$ no stuff coming in (all inward arcs are saturated)

$f^+(X) = c^+(X)$ all outgoing arcs at capacity "saturated"

<u>Pf:</u> if $a = (u,v)$ is in $\partial^+(X)$

then $\exists$ unsat (x,u) path P & no unsat (x,v) paths.

look at ~~the~~ path $P(\underbrace{u,v}_{a})$ not unsaturated

$\Rightarrow (u,v)$ saturated ...