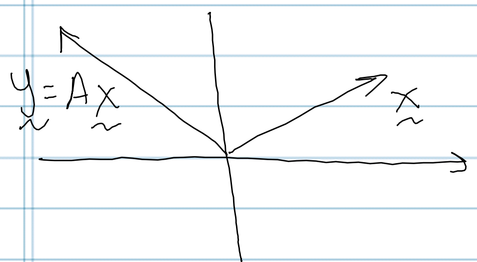
$\boxed{\text{SESSION 7}}$

Eigenvalues and eigenvectors:

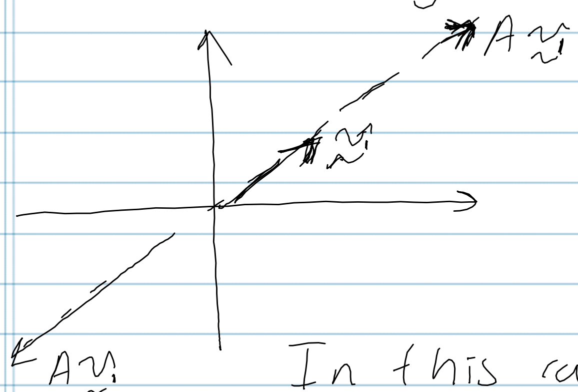
Let A be a 2×2 matrix. Let $\mathbf{x}$ be a 2×1 vector, and let $\mathbf{y} = A\mathbf{x}$



A is known as a linear transformation.

If $|\mathbf{x}| = |A\mathbf{x}|$, A is "orthogonal"

Two special vectors, the "eigenvectors" of A do not change direction when multiplied by A.



The length of $A\mathbf{v}_1$ is same as $\lambda_1 \mathbf{v}_1$. λ_1 is called an "eigenvalue"

Eigen = German for "innate"

In this case, there are 2 e-vals an 2 evecs.

Stretching or compression is determined by the eigenvalues. In general, an $N\times N$ matrix has at most N families of eigenvectors and at most N distinct eigenvalues.

Algebraically, $\mathbf{v}$ is an e-vec and λ an e-val if:

$$A\mathbf{v} = \lambda \mathbf{v}$$

If $\underset{\sim}{v}$ is an e-vec, then $c\underset{\sim}{v}$ is an e-vec as well, so there is an entire family of e-vec associated with eval λ.

Note that: $A\underset{\sim}{v} = \lambda \underset{\sim}{v}$
$\Rightarrow (A - \lambda I)\underset{\sim}{v} = 0$

The trivial solution $\underset{\sim}{v}$ preserves the equality, but it is uninteresting. We want to know what happens when $(A - \lambda I) = 0$.

There is a non-trivial solution iff $A - \lambda I$ is a singular matrix, i.e. $\text{Det}(A - \lambda I) = 0$. Solve this "characteristic equation" to fin λ. Once λ is known, replace into $(A - \lambda I)\underset{\sim}{v} = \underset{\sim}{0}$ and solve for the e-vec $\underset{\sim}{v}$

Ex: Find eigenvalues and eigenvectors of:

$$A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$$

Solve: $\text{Det}(A - \lambda I) = 0$

$$\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\det\left|\begin{pmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{pmatrix}\right| = 0$$

$$(4-\lambda)(-3-\lambda) - (-10) = 0$$

$$-(12 + 4\lambda - 3\lambda - \lambda^2) + 10 = 0$$

$$\boxed{\lambda^2 - \lambda - 2 = 0}$$ characteristic polynomial

$(\lambda-2)(\lambda+1)=0$

$\lambda_1=2, \quad \lambda_2=-1$

For eigenvectors:

1) $\begin{pmatrix} 2 & -5 \\ 2 & -5 \end{pmatrix} v_1 = 0 \quad \left(\begin{bmatrix} v_1 \\ \sim \end{bmatrix} \begin{bmatrix} v_2 \\ \sim \end{bmatrix} \right)$ → eigen value $=2$

$2v_1 - 5v_2 = 0$

$\Rightarrow v_2 = \frac{2}{5} v_1$

v_1 is arbitrary $(\neq 0)$

let $v_1 = 5$, then $v_2 = 2$

$\underset{\sim}{v_1} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ eigenvector associated with eigenvalue 2

2) For eigenvalue -1:

$\begin{pmatrix} 5 & -5 \\ 2 & -2 \end{pmatrix} \underset{\sim}{v_2} = \underset{\sim}{0}$ (Note: $(A-\lambda_2 I)v_2 = 0$)

$5v_1 - 5v_2 = 0$

$\Rightarrow v_1 = v_2$

v_1 is arbitrary $(\neq 0)$

$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, eigenvector associated with eigenvalue -1

Q.E.D.

If eigenvalues are distinct (don't repeat), then eigenvectors are linearly independent. So; any vector in ~~[illegible]~~ N-Dim. space can be written as a linear combination of the N eigenvectors.

$$\underset{\sim}{w} = c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2} + \dots + c_N \underset{\sim}{v_N}$$

for constants $c_1, \dots, c_N$