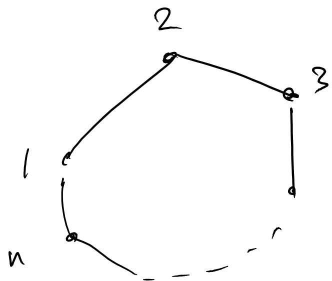
# Lecture 5: walks, components, connectedness

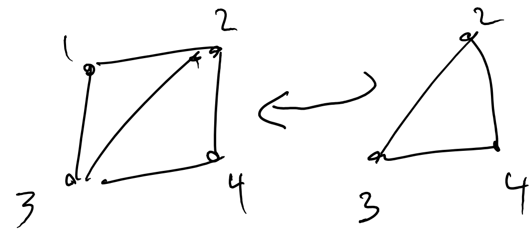
Tuesday, January 26, 2016 12:24 PM

<u>Last time:</u>

cycle graph $C_n$



n-cycle = subgraph isomorphic to $C_n$



is a 3-cycle

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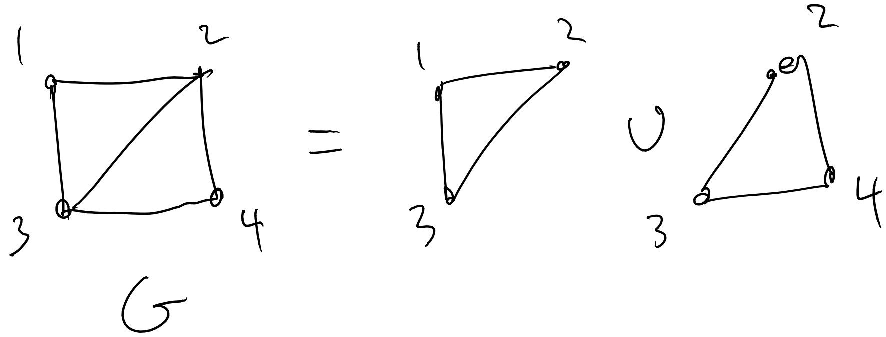
$G$ a graph, $H_1, H_2$ subgraphs of $G$

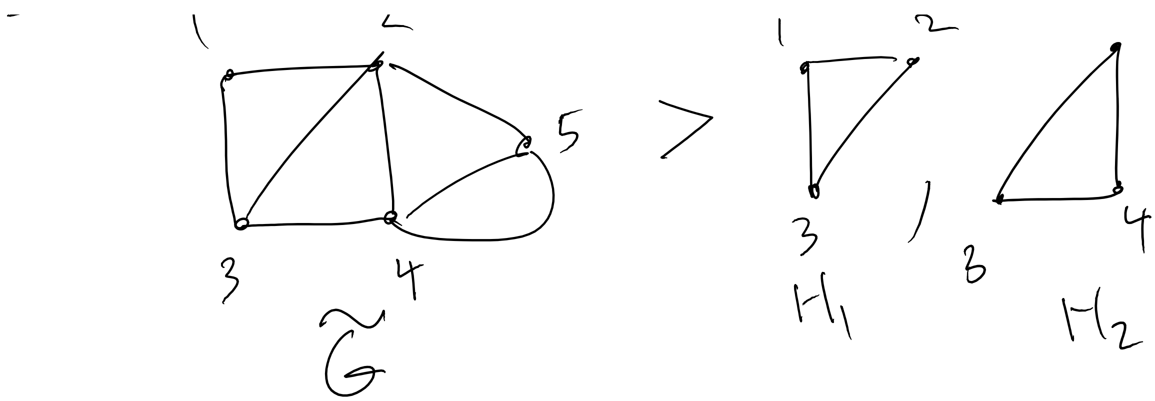
<u>Def</u> we say $G$ is union of $H_1$ & $H_2$, $G = H_1 \cup H_2$

if $V_{H_1} \cup V_{H_2} = V_G$, $E_{H_1} \cup E_{H_2} = E_G$



$G$

<u>ex:</u>




$$H_1 \cup H_2 = G$$

<u>Def</u> if $H_1, H_2 < G$, we define $H_1 \cap H_2$ to be the subgraph of $G$ described by

$$V_{H_1 \cap H_2} = V_{H_1} \cap V_{H_2}, \quad E_{H_1 \cap H_2} = E_{H_1} \cap E_{H_2}$$

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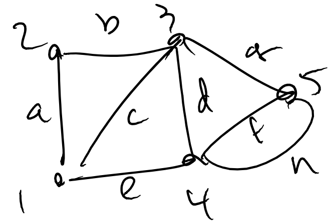
<u>Def</u> A walk in a graph $G$ is a sequence of alternate vertices & edges $\omega = v_1 e_2 v_2 e_3 v_3 e_4 \cdots e_n v_n$ $\quad v_i \in V_G$, $e_i \in E_G$

such that $e_i$ incident to $v_{i-1}$ & $v_i$




1 a 2 b 3 c 1 e 4 h 5 f 4

Note: if $G$ is simple, walk is determined by its list of vertices. we'll write $\omega = v_1 v_2 \cdots v_n$ in this case.

if $W = v_1 e_1 \cdots e_n v_n$ is a walk, $v_1 = v$ $v_n = w$
we'll call it a $(v, w)$ walk.

given a $(v,w)$ walk $W$ & a $(w,u)$ walk $W'$, can form a new walk $WW'$, a $(v,u)$-walk (concatenation)

via $W = v_1 e_2 v_2 \cdots e_n v_n$ $\quad W' = w_1 f_2 w_2 \cdots f_m w_m$

$v_1 = v$ $\quad v_n = w = w_1$ $\quad w_m = u$

$$WW' = v_1 e_2 v_2 \cdots e_n \underset{\substack{\shortparallel \\ v_n \\ \shortparallel \\ w_1}}{w} f_2 w_2 \cdots f_m \underset{\substack{\shortparallel \\ u}}{w_m}$$

define $W^{-1}$ as $v_n e_n v_{n-1} \cdots e_2 v_1$

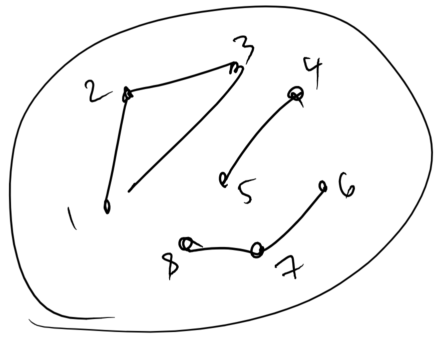
__Lemma__ if we define $v \sim w$ if and only if $\exists (v,w)$-walk then $\sim$ is an equiv. relation.

__Pf__ D.

$\Rightarrow$ can write $V_G = \bigcup_{i=1}^{r} V_i$ $\quad V_i$ are eq. classes.

__Def__ $G[V_i]$ are called the components of $G$

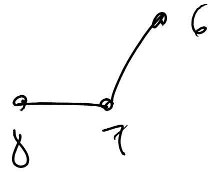
ex:




$G$

$V_1 = \{1,2,3\}$ $V_3 = \{6,7,8\}$

$V_2 = \{5,4\}$

$G[V_3] =$ 


Claim: $G = \bigcup G[V_i]$ ✓

Def: if $G = \bigcup H_i$, we say the union is disjoint if
$V_{H_i} \cap V_{H_j} = \emptyset$ for $i \neq j$

Def: if $G = \bigcup H_i$, we say union is edge disjoint
if $E_{H_i} \cap E_{H_j} = \emptyset$ all $i, j$

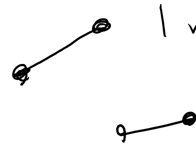
(disjoint $\Rightarrow$ edge disjoint)

Notation: if $V_G = \bigcup_{i=1}^{r} V_i$ equ classes, we set
$c(G) = r$

Def: $G$ is connected if $c(G) = 1$.

Def: $G$ is called $k$-regular if each vertex $v \in V_G$
has $\deg(v) = k$.

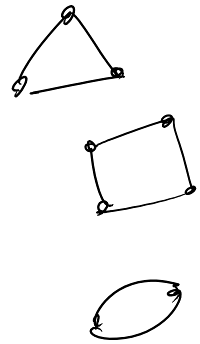
**Def** G is ... has $dg(v) = k$.



1 regular graph.

**Prop** if $G$ is a simple connected 2-regular graph, then $G \cong C_n$ some $n$.

2 regular graphs

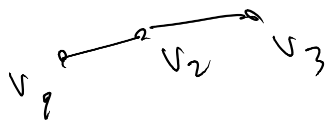


$C_n$

**Pf:** choose vertex $v = v_1$

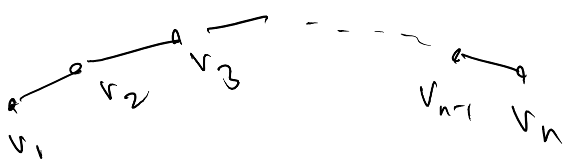

$v_1$ adjacent to 2 vertices
pick one: $v_2$


$v_2$ adjacent to $v_1$ & one other: $v_3$




inductively get

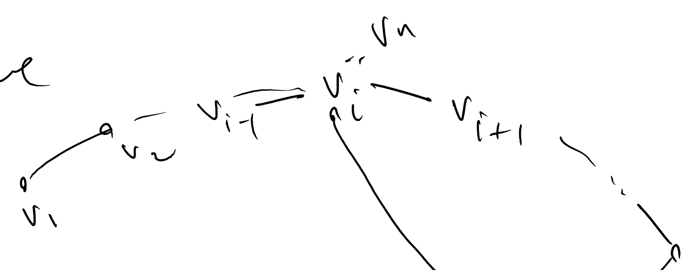
$W_n =$

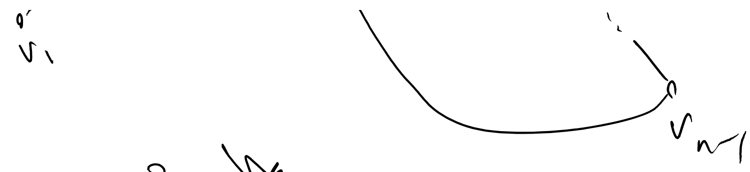



$G$ finite, eventually have $v_n = v_i$ some $i$.

know $v_i \neq v_{n-2}$, suppose $v_n \neq v_1$

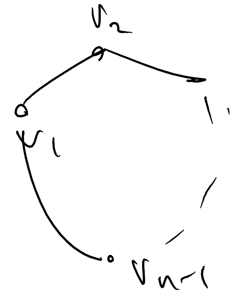
then we have






$\Rightarrow \deg v_i \geq 3$ $\forall i$.

$\Rightarrow v_n = v_1 \Rightarrow$




define isomorphism

$$C_{n-1} \longrightarrow G$$


$i \longmapsto v_i$ ✓.

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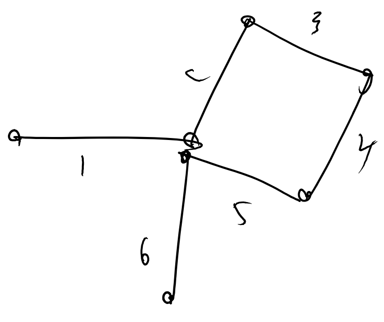
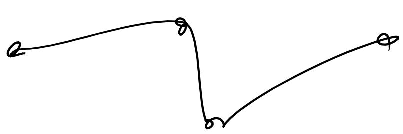
__Def__: A __trail__ is a walk whose edges are distinct




__Def__ A __path__ is a walk whose vertices are distinct.



__Lem__ In a graph $G$, $\exists$ a $(u,w)$-walk $\Longleftrightarrow \exists$ $(u,w)$-path.

**Lem** In a graph $G$, $\exists$ a $(u,w)$-walk $\Leftrightarrow$ $\exists$ $(u,w)$-path.

**Def** if $G$ is a graph, $e \in E_G$ is a bridge if $c(G) < c(G-e)$

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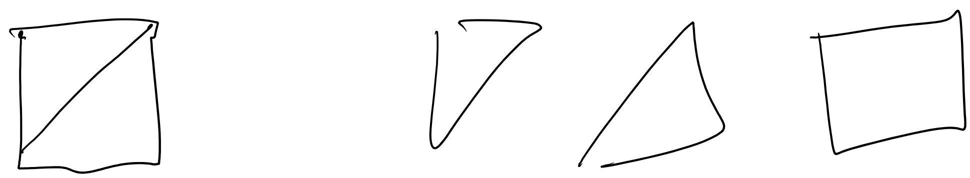
**Conjecture** (Szerkers-Seymour)

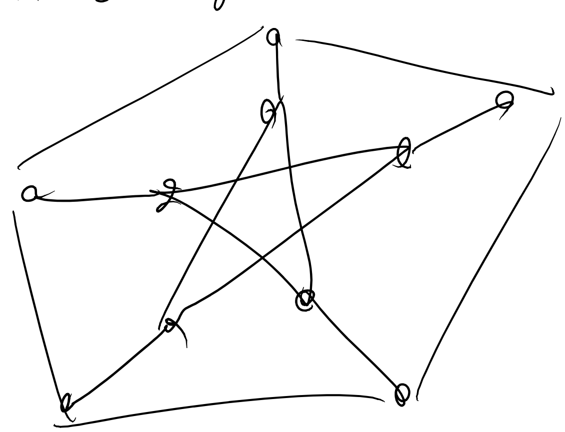
every simple bridgeless graph has a collection of cycles such that every edge is in exactly 2 cycles.



"cycle double cover conjecture"

can reduce problem to the case of **snarks**

snark = connected simple 3-regular graph which is not 3-edge colorable.

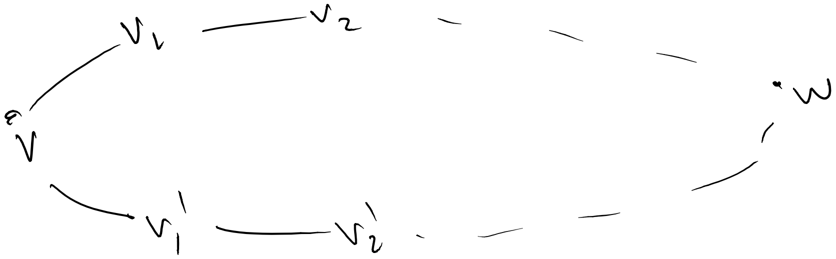


**Def** A forest is a simple graph with no cycles.

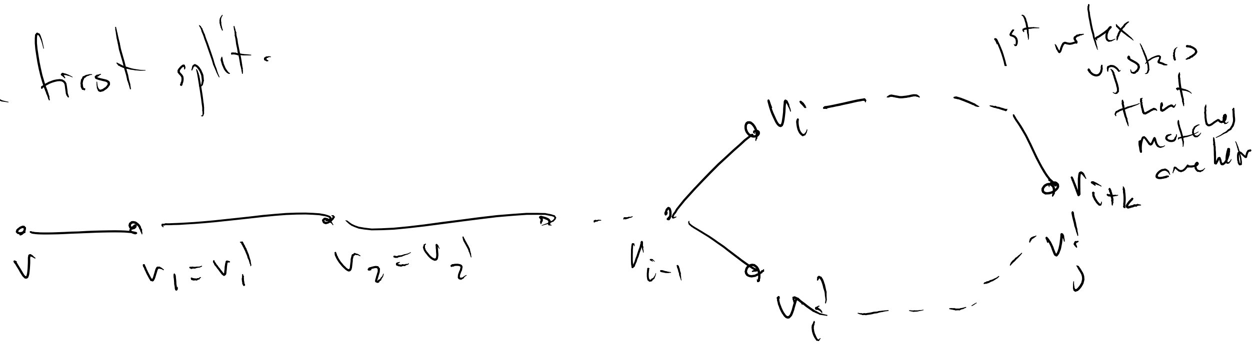
**Def** A tree is a connected forest.

**Thm** A graph is a tree if it is connected and if there is a unique path between any 2 vertices.

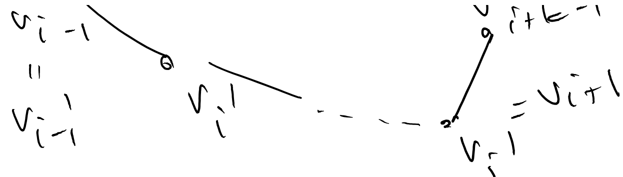
**Pf:** if there are two paths v to w




same first split.




$\Rightarrow$




is a cycle in $G$.