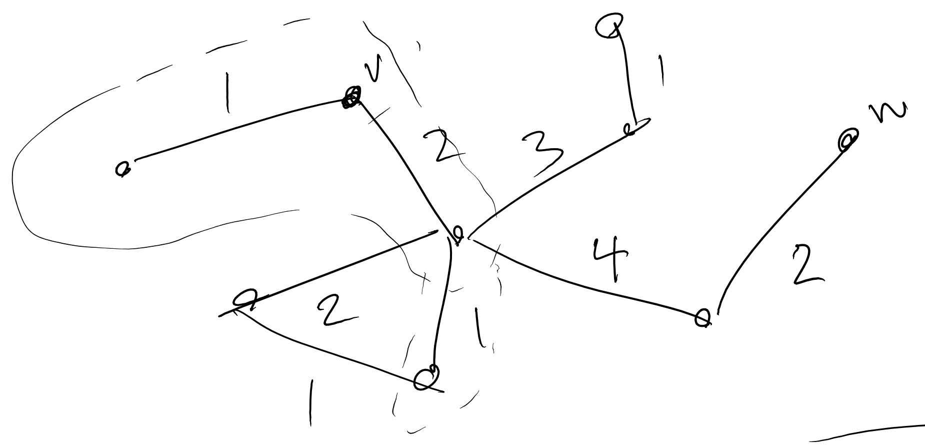
# Lecture 7: Dijkstra's algorithm, part 2; Bonds, part 1

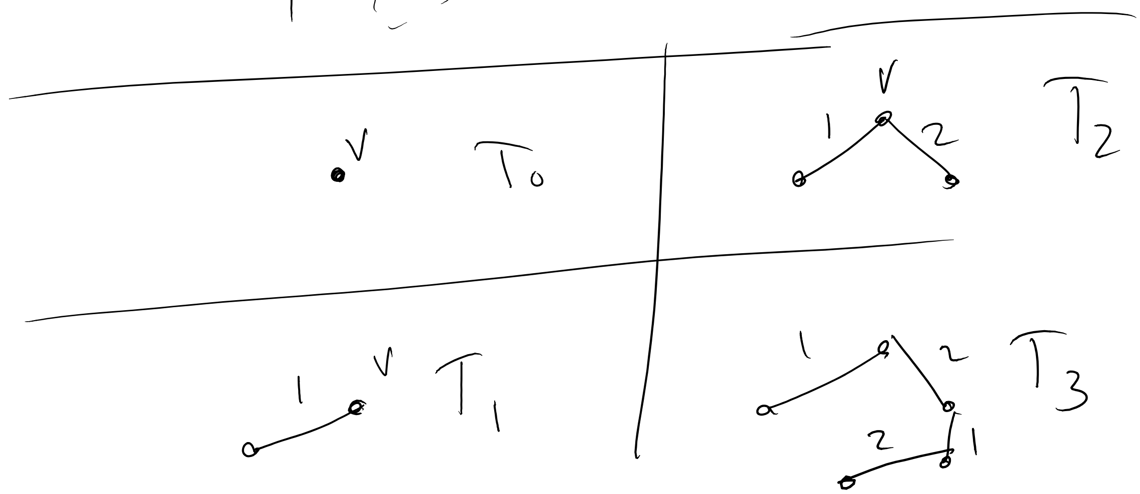
Tuesday, February 2, 2016 12:38 PM

last time: Dijkstra's algorithm.

$G$ graph $w : E_G \to \mathbb{R}_{>0}$ ($G$ connected)

$\rightsquigarrow$ from a given vertex $v$, a spanning tree, minimizing distances from $v$ to every other vertex



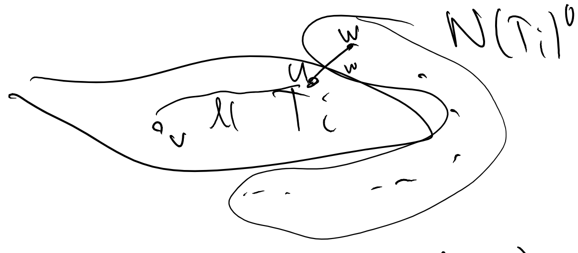

Formally:

- Set $T_0$ = subgraph w/ vertex set $\{v\}$ no edges.
- Assume have already constructed $T_i$,

choose $w \in N(T_i)^0 = \{u$ adj to some $u' \in V_{T_i}$ but not in $T_i\}$

and $u \in T_i$ so that $u$ adj to $w$,

& $l((v,u)$-path in $T_i) + w($edge between $u$ & $w)$ is minimal




Set $T_{i+1} = T_i + (uw$ edge$)$

Claim: 1 $T_i$ is a tree at each step.

2 new edge added to get $T_{i+1}$ (edge $e$ from $u$ to $w$) makes $l(v,u$ path$) + w(e) = d(v,w)$

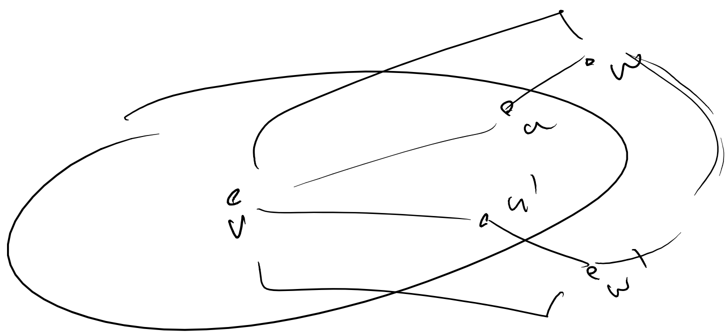
Pf of 1: $\deg_{T_{i+1}} w = 1 \Rightarrow w$ can't be part of a cycle in $T_{i+1}$ if $i+1$ is minimal s.t. $\exists$ cycle

2: Notation $P_{v,u}$ = minimal path in $T_i$ found previously.

have by assumption $l(P_{v,u}) + w(e_{uw})$ minimal.

assume that $P'$ any path from $v$ to $w$ in $G$

want to show: $\ell(P') \geq \ell(P_{v,u}) + w(e_{uw})$




$\triangle$.

<u>Observation</u>: $v(T_i) = i+1 \qquad e(T_i) = i$

<u>Conclusion</u>: every connected graph $G$ has a spanning subtree $T$ w/ $e(T)+1 = v(T)$

<u>Pf</u> define $d \equiv 1$.

<u>Cor</u> $T$ is a tree $\iff$ $T$ connected & $v(T) = e(T)+1$

<u>Pf</u>: let $T' \subset T$ be a spanning tree as above.

$\Leftarrow$ $v(T') = v(T)$ by construction, $e(T') = v(T') - 1$

$= v(T) - 1$

$= e(T)$

$\Rightarrow E_T = E_{T'} \Rightarrow T = T'$. $\triangle$

<u>Interpretation</u>: Trees are minimal connected graphs

... a spanning

Interpretation: Trees are minimal connec

example prop: $G$ is connected $\Leftrightarrow$ $G$ has a spanning subtree.

Lem If $T$ is a spanning subtree in $G$, and $e \in E_G \setminus E_T$ then $T+e$ contains a unique cycle.

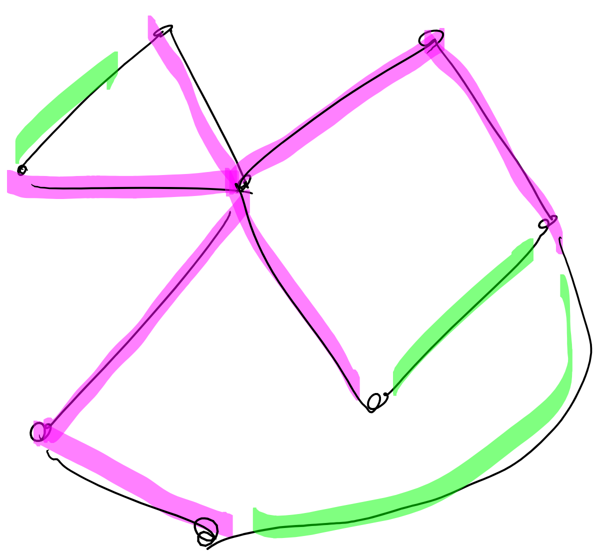
Pf. in $T+e$, any cycle must involve $e$


rest of the cycle must consist of a path from $v$ to $w$.

but path is in $T$

$\Rightarrow$ path unique. ↯

"Fundamental cycle of $G$ w.r.t. to $T$ & $e$"

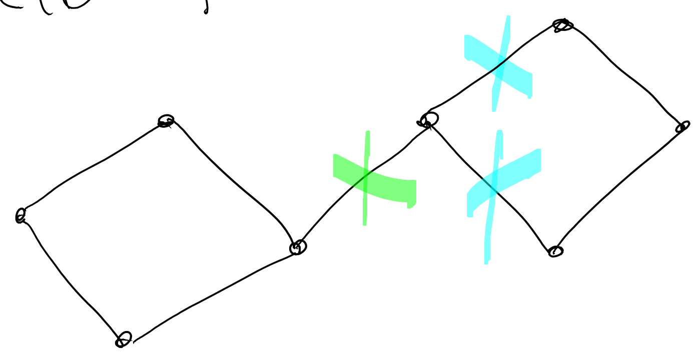


spanning subtree

cotree.

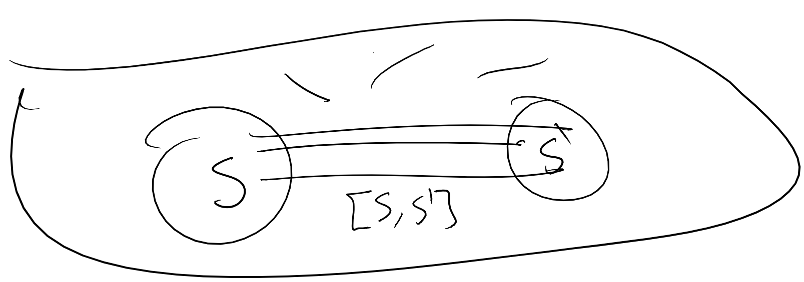
Def cotree $\overline{T}$ is a set of edges of the form $E_G \setminus E_T$
$T$ a spanning subtree.

Def A bond* = min'l collection of edges $E \subset E_G$ s.t.
$c(G-E) > c(G)$



eg: a bond w/ 1 elmt is a bridge.

Def If $G$ a graph, $S, S' \subset V_G$,
$[S,S'] = \{$ edges $e \in E_G \mid e$ incident to vertices in both $S$ & $S'$ $\}$




Def An edge cut is a set of edges of the form $[S, \overline{S}]$ $\quad \overline{S} = V_G \setminus S \quad$ both $S, \overline{S} \neq \emptyset$

Def A bond is a minimal edge cut.

**Def** A bond is a minimal edge cut. (nonempty)

**Exercise** bond* = bond