Lecture 8: Cuts and connectivity

Thursday, February 4, 2016 12:34 PM

Last time:

Bonds = min'l edge cut / min'l edges to make more components

Various notions of connectivity (G connected)

$\kappa'(G)$ = min'l # of edges whose removal makes G (trivial or) disconnected

$\kappa(G)$ = min'l # of vertices whose removal makes G (trivial or) disconnected.

Trivial today means: 0 or 1 vertex

Vertex cut: is a collection of vertices $W \subset V_G$ s.t. $c(G-W) > c(G)$

Set of separating vertices: vertices $W \subset V_G$ s.t. $G = G_1 \cup G_2$ edge disjoint union s.t. $V_{G_1} \cap V_{G_2} = W$.

$G - W = G[V_G \setminus W]$

Thm $\kappa(G) \leq \kappa'(G) \leq \delta(G)$

$\delta(G)$ = minimal degree of a vertex degree of a vertex in G

min'l # of vertices to make disconnected / trivial

min'l # edges to make disconnected

trivial in this context = • (or $\emptyset$)

Proof: if deg v = d then say $e_1, \dots, e_d$ edges inc. to v.

$G - \{e_1, \dots, e_d\}$ is disconnected or trivial ✓

to show $\kappa(G) \le \kappa'(G)$:

suppose that removal of some set of $l = \kappa'(G)$ edges disconnects or makes G trivial. want to show some set of at most l vertices when removed makes G disconnected.

Induct on l

$l = 0$

Let G be a graph w/ $\kappa'(G) = l$, let $F \subset E_G$ be a min'l disconnecting set of edges. choose $e \in F$

In $G - e$, $\kappa'(G-e) = l-1$, $\kappa(G-e) \le l-1$ (induction)

can find vertices $v_1, \dots, v_m \in V_{G-e}$ $m \le l-1$

s.t. $(G-e)-\{v_1,\dots,v_m\}$ trivial or disconnected.

consider $G-\{v_1,\dots,v_m\}$.

If e is in $G-\{v_1,\dots,v_m\}$ (then e not incident to any v_i's)

$$(G-\{v_1,\dots,v_m\})-e = (G-e)-\{v_1,\dots,v_m\}$$

= trivial or disconnected

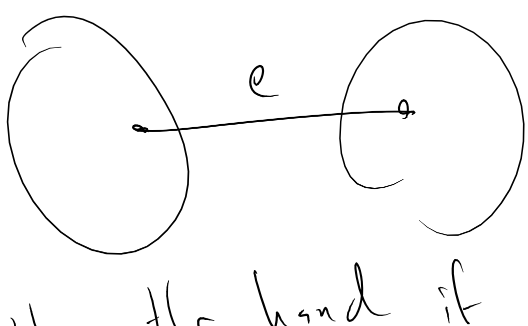
$\Rightarrow \kappa'(G-\{v_1,\dots,v_m\}) \le 1 \Rightarrow \kappa(G-\{v_1,\dots,v_m\}) \le 1$ if claim is true for $l=1$

If e is not in $G-\{v_1,\dots,v_m\}$

$G-\{v_1,\dots,v_m\} = (G-e)-\{v_1,\dots,v_m\}$

= triv. or disc. $\Rightarrow \kappa(G) \le m \le l-1 < l = \kappa'(G)$

Need only show true if $\kappa'(G)=1$



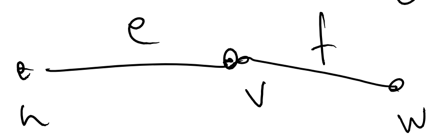
if $G-e$ trivial then $G \in \{ \bullet\!-\!\bullet, \circ \}$

on the other hand if $G-e$ disconnected nontrivial

G connected, $G-e$ nontrivial $\Rightarrow$ at least an edge adj

G connected, ...

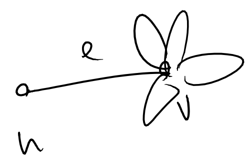
at least an edge adj to e



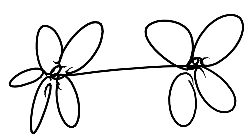
either e is adjacent to an edge which is not a loop.

in this case, remove v, $G - v \ni u, w$ & no (uv)-paths in $G-v$

if every edge adj. to v is a loop



make same argument w/ u,



D.

Menger's Theorem(s)

Def if $u, v \in V_G$, a uv edge cut is a set of edges $E' \subset E_G$ s.t. $\nexists$ uv-path in $G - E'$.

Def $\kappa'(u,v)$ min'l # of edges in a uv edge cut.

easy to check: $\kappa(G) = \min\{\kappa'(u,v) \mid u,v \in G\}$ if G is not trivial.

if G is not trivial.

<u>Thm</u> If G is conected graph then

$K'(u,v)$ = max size of a collection of pairwise edge - disjoint (u,v)-paths