Lecture 9: connectivity, part 2

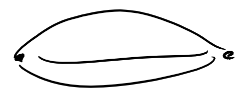
Tuesday, February 9, 2016 12:37 PM

disconnecting a graph

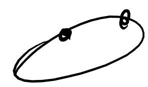
$\kappa(G)$

$\kappa'(G)$

independent paths



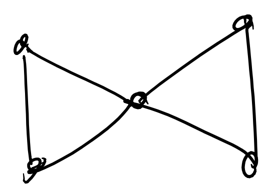
cycles



if G (simple) has no bridges

then every edge lies on a cycle.
or bigon

A vertex $v \in V_G$ is a <u>cut vertex</u> if $c(G-v) > c(G)$



<u>Prop</u>: Suppose G is connected, has at least 3 vertices and no cut vertices, then every pair of vertices lie on a cycle.

<u>Pf</u>: Define distance between vertices = length of a min'l path between them

 dist = 1

Choose vertices $u, v \in V_G$, induct on distance between them.

dist $= 1$

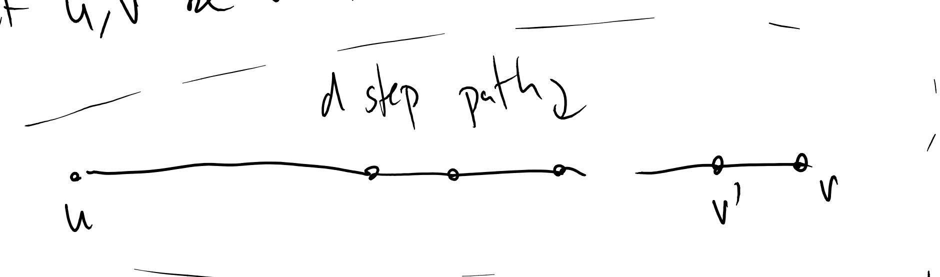
remove e, $G - e$ is connected since $\kappa'(G) \geq \kappa(G) \geq 2$

$\Rightarrow$ e not a bridge.

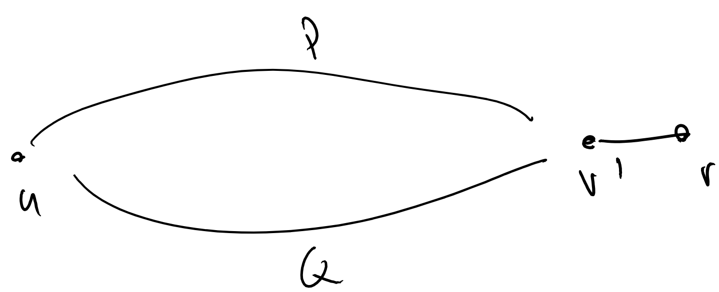
$\Rightarrow$ e on a cycle $\square$

suppose true for vertices $\leq d-1$ apart.

let u, v be vertices distance d.

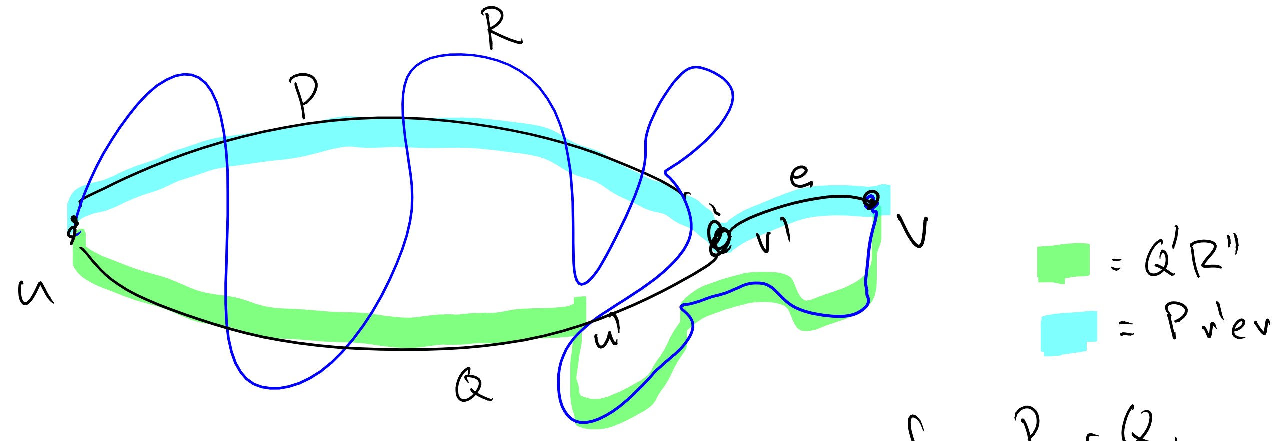


$\Rightarrow$ u & v' are $d-1$ apart $\Rightarrow$ $\exists$ disj. paths P, Q from u to v'



now consider $G - v'$. This is connected, $u, v \in V_{G-v'}$

so can find a (u, v)-path R in $G - v'$



<u>Case 1</u>: if R happens to be disjoint from P or Q,

done: 2 paths are R & $Pv'ev$

(say R disj from P)

<u>Case 2</u>: R hits both. suppose in going from u to v

it hits Q last at some vertex u'

$R = R'R''$ where R' is a (u, u')-path

R'' is a (u', v)-path

by construction, R'' is int. disjoint from P & Q.

u' along Q, so can write $Q = Q'Q''$

Q' is a (u, u')-path,

Q'' is a (u', v')-path

now, new int. disj. paths are

$\square$

now, new int. disj. paths are
$P v' e v$ & $Q' R''$. $\square$

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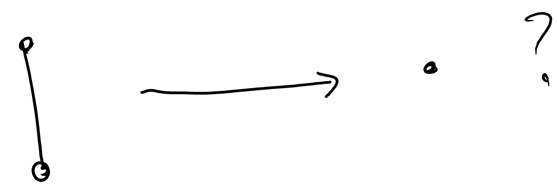
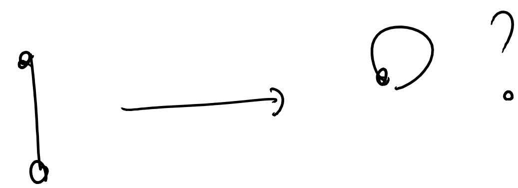
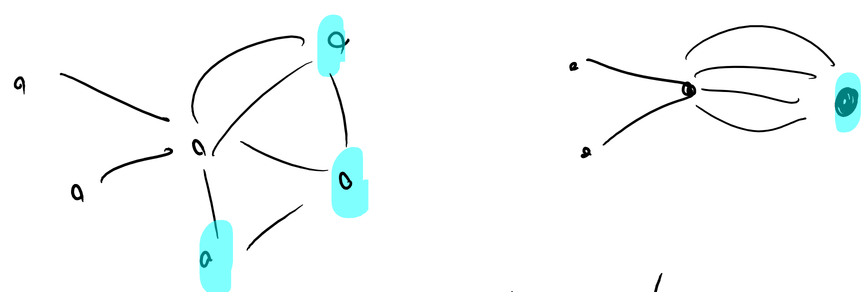
maps between graphs (why we don't generally talk about it)





Given a graph G, $A \subset V_G$, define G/A
"shrinking A to a vertex"



Def G/A is the graph w/
$V_{G/A} = (V_G \setminus A) \cup \{a\}$

$$E_{G/A} = E_G \setminus [A,A]$$

$$\psi_{G/A}(e) = \begin{cases} \psi_G(e) & \text{if } e \in [\bar{A},\bar{A}] \\ va & \text{if } \psi_G(e) = va' \text{ some } a' \in A. \end{cases}$$

<u>Thm</u> (Menger) G is a connected graph, u,v nonadjacent
then $\kappa(u,v) = \max \#$ of pairwise disjoint (u,v)-paths.

<u>Defs</u>: $\kappa(u,v) = \min \#$ of vertices we need to remove so that there are no (u,v) paths.

(only defined if u,v nonadjacent)

<u>Note</u>: $\kappa(G) = \min\{\kappa(u,v) \mid u,v \in V_G\}$ if $v(G)-1 \neq \kappa(G)$

<u>Def</u>: $p(u,v) = \max \#$ mutually disj. paths from u to v.

<u>Pf</u> WTS $p = p(u,v) = \kappa(u,v) = \kappa$

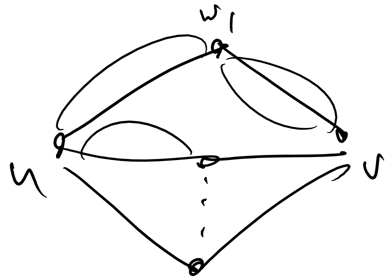
Induct on $e(G)$.

Base case: $e(G) = 2$ ✓. $\kappa = p = 1$.

In general <u>Case 1</u>: every edge in G is incident to either v or u.

In gen... — to either ...

in this case, every path from u to v has length 2



$P_1, P_2, \ldots, P_\ell$ indep. paths.

$P_i = u e_i w_i f_i v$

w_i all distinct.

$\Rightarrow$ if P_i's are a max'l family then $p = \ell$

& also $G - \{w_1, \ldots, w_\ell\}$ has no (u,v) paths.

$\Rightarrow K \leq \ell = p$