Recall: If we are given two vertices, $v, w \in V(G)$ in same component, we define $d(v,w)$ = minimal length of path from v to w.

minimal paths are also called geodesics.

Def $\text{diam}(G) = \max \{ d(v,w) \mid v,w \in V(G) \}$

(trivial = $\bullet$)

Theorem If G is a nontrivial connected graph and $u \in V(G)$, v is as far as possible from u (i.e. $d(u,v)$ maximum) over all possible v's

then v is not a cut vertex.

Aside: Suppose G is connected. If $v \in G$ is a cut vertex then $G-v$ is disconnected, so can find $u, w \in G-v$ in different components

$\Rightarrow$ every path from u to w in G must include v.

Conversely, if we have two vertices u, w in a connected graph such that every $u-w$ path passes through some other fixed vertex v then v must be a cut vertex.

Theorem A vertex $v \in V(G)$, G connected is a cut vertex if and only if $\exists\, u, w \in V(G)$ s.t. v lies on every $u-w$ path.
$u, w \neq v$

Pf of thm ☆

Suppose u fixed, v chosen such that $d(u,v)$ maximum.

suppose (arguing by contradiction) that v is a cut vertex

pick w on comp. of $G-v$ which doesn't include u.

by previous arguments, every u-w path includes v.

$\Rightarrow d(u,w) > d(u,v)$ contradicts maximality.

<u>Cor</u>: if we choose $u,v \in V(G)$ G connected s.t.

$d(u,v) = \operatorname{diam}(G) \Rightarrow u$ & v are both not cut vertices.

So every connected graph contains at least two non-cut vertices.

<u>Note</u>: in a tree - every edge is a bridge $\Rightarrow$ every vertex is either a cut vertex or degree 1. (leaf).

Cor $\Rightarrow \exists$ at least two vertices not cut vertices

$\Rightarrow$ at least 2 leaves.

Def A connected graph is called <u>nonseparable</u> if it has no cut vertices.

Def A graph G is called disconnected if we can find subgraphs $H_1, H_2 \subset G$ such that

$$V(G) = V(H_1) \cup V(H_2) \qquad E(G) = E(H_1) \cup E(H_2)$$

and $V(H_1) \cap V(H_2) = \emptyset$.

Prop ☺ A graph G is connected if and only if it is not disconnected.

Pf: Suppose G is connected. want to show that G is not disconnected. Argue by contradiction - suppose G is disconnected, i.e. $\exists$ H_1, H_2 as above. because $V(H_i) \neq \emptyset$ can find $v_i \in V(H_i)$ $i = 1, 2$. now, there is a path from v_1 to v_2

path starts in H_1 & ends in H_2.

let u in the path be the first vertex in H_2 and u' the vertex in path just before it. $e = uu'$ the edge in path connecting them. $e \in E(G) = E(H_1) \cup E(H_2)$ so

$e \in E(H_i)$ some i. say $i = 1$

$\Rightarrow$ vertices incident to e are also in H_1

$\Rightarrow u, u' \in V(H_1) \quad u \in V(H_2)$

$V(H_1) \cap V(H_2) \neq \emptyset$ contradiction.

Conversely,

Suppose G is not disconnected. why is G connected?

Given v, w, want to show there is a walk from v to w

Let V_1 be the set of vertices s.t. can walk from v

$$V_1 = \{u \in V(G) \mid \exists \text{ a } v-u \text{ walk}\}$$

$$V_2 = \{u \in V(G) \mid \nexists \text{ a } v-u \text{ walk}\}$$

$H_i = G[V_i]$ know $V_1 \neq \emptyset$. since $v \in V_1$.

$V(H_i) = V_i \Rightarrow V(H_1) \cap V(H_2) = \emptyset$

$V(H_1) \cup V(H_2) = V(G)$

$E(G) = E(H_1) \cup E(H_2)$?

this is the statement that every edge must connect two vertices in V_1 or two vertices in V_2 but <u>not</u> a vertex in V_1 and one in V_2.

true since if you can walk to v_1 from v and v_1 adjacent to v_2 then can walk from v to v_2.

$\Rightarrow$ if $V_2 \neq \emptyset$ would have G is disconnected. ✓

$\Rightarrow V_2 = \emptyset. \Rightarrow V_1 = V(G) \Rightarrow$ can walk anywhere from v.

Def A graph G is separable if $\exists$ subgraphs H_1, H_2 such that $E(H_1) \cup E(H_2) = E(G)$

$$V(H_1) \cup V(H_2) = V(G)$$

$$E(H_1) \cap E(H_2) = \emptyset$$

and $V(H_1) \cap V(H_2)$ has at most one element.

if G is connected then will have $\#(V(H_1) \cap V(H_2)) = 1$.

Prop. A connected graph G is separable if and only if it has a cut vertex. ($\Leftrightarrow$ G is nonseparable iff G is not separable)

Pf sketch of idea: if G is separable, choose H_1, H_2 as above, then $v \in V(H_1) \cap V(H_2)$ is cut vertex.

and conversely, if v is a cut vertex, say H_1', H_2' subgraph of $G - v$, as in

Set $H_i = G[V(H_i') \cup \{v\}]$

Theorem If G is a graph with at least 3 vertices, then G is nonseparable if and only if every two vertices lie on a common cycle.

<u>Pf:</u> Suppose every two vertices lie on a common cycle.

Suppose v is a cut vertex (arguing by contradiction)

choose u, w on diff. components of $G - v$.

by assumption, $\exists$ cycle in G containing u, w $\Rightarrow$

there are two mutually disjoint $u-w$ paths.

$\Rightarrow$ v not on both paths $\Rightarrow$ $\exists$ $u-w$ path not

involving v $\Rightarrow$ contradiction.

Suppose conversely that G is nonseparable.

note G has no bridges since G is not •—•

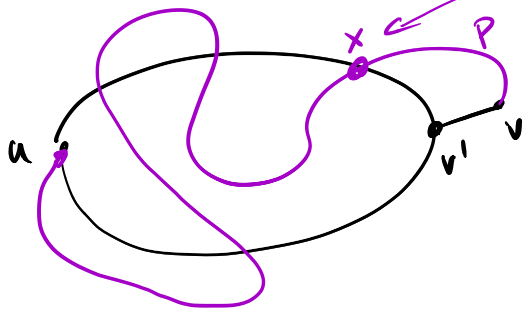
if there are pairs of vertices not on a common cycle

choose u, v not on a common cycle w/ $d(u,v)$ as small

as possible. note $d(u,v) \geq 2$ since if u adjacent to v

then $e = uv$ not on a cycle $\Rightarrow$ e is a bridge.

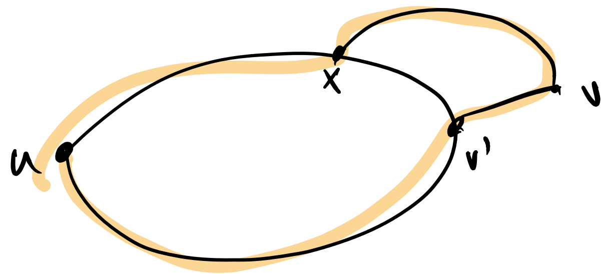
first pt P hits cycle.



v' not a cut vertex

$\Rightarrow$ $\exists$ path P $v - u$

not involving v'


x
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