Equivalence Relations

If S is a set, a <u>relation</u> on S is a subset of $S \times S$.

If $R \subset S \times S$ is a relation, we say that a is R-related to b

(and write aRb) to mean $(a,b) \in R$.

<u>Ex</u>: $S = \mathbb{R}$ $\quad R = \{(x,y) \mid y = x^2\}$

$2R4$. $\quad 3 \not R 7$ (not related)

$4 \not R 2$ $\quad 1R1$

<u>Ex</u>: $S = \mathbb{Z}$ $\quad R = \{(x,y) \mid |x-y| = 1\}$

$1R2 \quad 2R1 \quad 7R8$

$xRy \iff yRx$

<u>Ex</u>: $S = \mathbb{Z}$ $\quad R = \{(x,y) \mid x \leq y\}$

<u>Ex</u>: $S = \{\text{graphs}\}$ $\quad R = \{(G_1, G_2) \mid G_1 \cong G_2\}$

isomorphic to

<u>Def</u> A relation $\sim \subseteq S \times S$ is an <u>equivalence relation</u> if it satisfies the following conditions:

1) $x \sim x$ all $x \in S$ (reflexivity)

2) $x \sim y \Rightarrow y \sim x$ all $x, y \in S$ (symmetry)

3) $x \sim y, y \sim z \Rightarrow x \sim z$ all $x, y, z \in S$ (transitivity)

Ex: G a graph, $S = V(G)$, define $v \sim w$ if there exists a v-w walk in G.

this is an equivalence relation.

Def If S is a set, a partition of S is a collection of subsets $P \subset \mathcal{P}(S)$, each element of P nonempty such that:

1) $\bigcup_{p \in P} p = S$ 2) if $p, q \in P$, $p \neq q$ then $p \cap q = \emptyset$

ex: $S = \mathbb{Z}$ $p = \{\text{even \#s}\}$ $q = \{\text{odd \#s}\}$

$P = \{p, q\}$ is a partition of $\mathbb{Z}$.

Def If S is a set, $\sim$ an equivalence relation, $a \in S$

$[a] = [a]_\sim = \{b \in S \mid a \sim b\}$ "the equiv. class of a"

Theorem (Fundamental theorem of equivalence relations)

Let S be a set.

If $\sim$ is an eq. relation on S, define

$$P_\sim = \{ [a]_\sim \mid a \in S \}.$$

And if P is a partition, define

$$\sim_P = \{ (a,b) \in S \times S \mid a,b \in p \text{ some } p \in P \}.$$

Then: $P_\sim$ is a partition whenever $\sim$ is an eq. rel

and $\sim_P$ is an eq. rel, whenever P is a partition

and $\quad P_{\sim_P} = P \qquad \sim_{P_\sim} = \sim$

i.e. if S is any set

$$\begin{tikzcd} \{\text{partitions of } S\} \arrow[r, shift right, "{P \mapsto \sim_P}"'] & \{\text{eq. rels on } S\} \arrow[l, shift right, "{P_\sim \mapsfrom \sim}"'] \end{tikzcd}$$

and these give mutually inverse bijections.

Pf: Fix S a set,

Let $\sim$ be an eq. rel. First: want to show $P_\sim$ is a partition

$$P_\sim = \{ [a]_\sim \mid a \in S \}$$

Aside:

$S = \mathbb{Z}$ $\quad \sim = \{ (a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a-b \text{ is even} \}$

$[0] = \{ b \mid 0-b \text{ even} \} = \{ \text{even \#s} \}$

$[1] = \{ b \mid 1-b \text{ even} \} = \{ \text{odd \#s} \}$

$[2] = [0] = [-8]$

$[1] = [5] = [-11]$ etc.

$P_\sim = \{ [a] \mid a \in \mathbb{Z} \} = \{ [0], [1] \}$

$= \{ p, q \}$

$p = \{ \text{even \#s} \}$ $\quad q = \{ \text{odd \#s} \}$

In general, want to check

0) $[a] \neq \emptyset$ $\qquad$ 1) $\bigcup_{a \in S} [a] = S$

2) $[a] \cap [b] = \emptyset$ if $[a] \neq [b]$

0) $a \in [a]$ so nonempty ✓

1) $\forall\, b \in S$, $b \in [b] \subset \bigcup_{a \in S} [a]$

so $S \subset \bigcup_{a \in S} [a]$ $\quad \bigcup [a] \subset S$ trivially.

2) Suppose $x \in [a] \cap [b]$. we'll show $[a] = [b]$.

Let's check $[a] \subset [b]$.

if $y \in [a]$ then $a \sim y$

but $x \in [a] \Rightarrow a \sim x \overset{\text{sym}}{\Rightarrow} x \sim a$

trans & symmetry $\Rightarrow x \sim y \overset{\text{sym}}{\Rightarrow} y \sim x$

but $x \sim b$, $\Rightarrow y \sim b \Rightarrow y \in [b]$

QED $\square$

Why is $\sim_p$ an eq rel.

$a \sim_p a$ is the statement that

$a, a \in p$ some $p \in P$

follows from $\bigcup_{p \in P} p = S$

$\Downarrow$ $a \in p$ some p.

$a \sim_p b$, why is $b \sim_p a$? ✓

$a \sim_p b \Leftrightarrow a, b \in p$ some $p \in P \Leftrightarrow b, a \in p$ some $p \in P$.

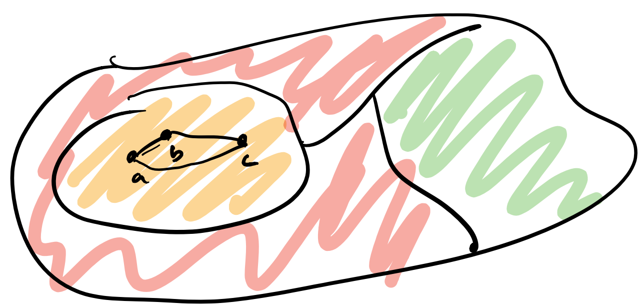
$a \sim_{\underline{P}} b$, $b \sim_{\underline{P}} c$, why is $a \sim_{\underline{P}} c$?

if $a \sim_{\underline{P}} b \Rightarrow \exists\, p \in \underline{P}$ s.t. $a, b \in p$

$b \sim_{\underline{P}} c \Rightarrow \exists\, q \in \underline{P}$ s.t. $b, c \in q$

but $b \in p$ & $b \in q \Rightarrow p \cap q \neq \emptyset \Rightarrow p = q$.

$\Rightarrow a, c \in p = q. \Rightarrow a \sim_{\underline{P}} c.$



If G is a graph

<u>Def</u> ~~$v \sim w$ for $v, w \in V(G)$ if v is adjacent to w.~~

~~symmetric, but not an equiv rel.~~

<u>Def</u> $v \sim w$ if $\exists$ v-w walk in G.

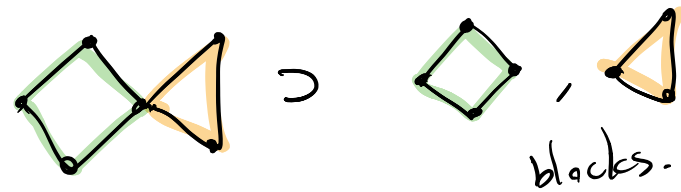
is an equiv. rel.

equivalence classes correspond to components of G.

Recall: A connected graph G is <u>nonseparable</u> if it has no cut vertices.

Def A block in a connected graph is a maximal connected nonseparable subgraph.

ex:



blocks.

Lemma: Define for $e, f \in E(G)$ $e \sim f$ if e & f lie on a common cycle or $e = f$. Then this is an eq. relation.

Theorem: Two edges e, f are in the same block of a connected graph G if and only if $e \sim f$ as above.

$\boxed{\text{Announcement}}$ office hours today are <u>delayed</u> to 5-6pm.