Graph Theory Practice Sheet for Midterm 2

This sheet is not meant to be exhaustive, but rather as a supplement to the problems from the homework since the last exam.

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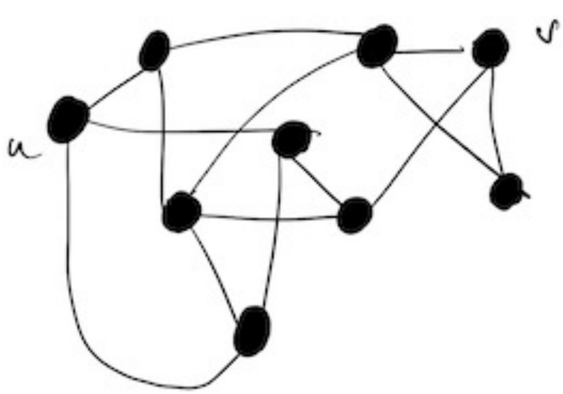
proof

1. This is a problem in the direction of Vizing's Theorem. Show that for a graph G, you can always color it using at most $2\Delta - 1$ colors. As a hint, you should think about the simple vertex coloring algorithm and how it worked.
2. This problem is the direction of Brook's Theorem. Suppose that G is a graph, v is a cut vertex and $G_1, G_2, \ldots, G_k$ are the components of $G - v$. Show that if $\chi(G_i)$ is less than $\boxed{\Delta(G)}$ [redacted] for each [redacted] i, then we will also have $\chi(G) \leq \Delta + 1$. χ $\leq$ Δ
3. Can you draw a graph with $\chi(G) = 4$ and with the graph containing no triangles? If you can, do it. If not, say why not.
4. Draw a graph with $\kappa(G) = 2, \lambda(G) = 2$ and $\delta(G) = 3$ (or show no such graph exists).

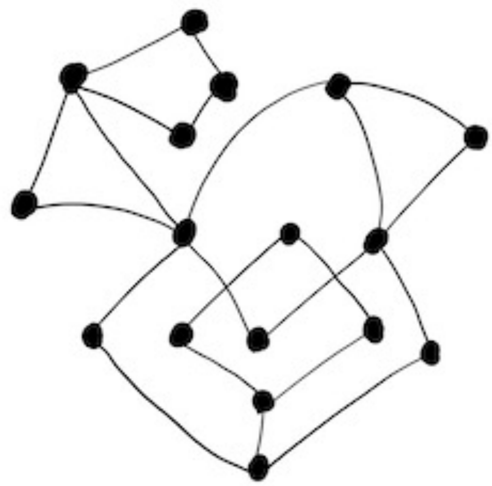
Can you draw?

5. In the graph shown below, exhibit a minimum $u - v$ vertex cut and a minimal $u - v$ vertex cut which isn't minimum. How can you tell that your minimum vertex cut is actually minimum?

min'm vs min'l



6. Find all cut vertices and blocks in the graph below:



what's a block

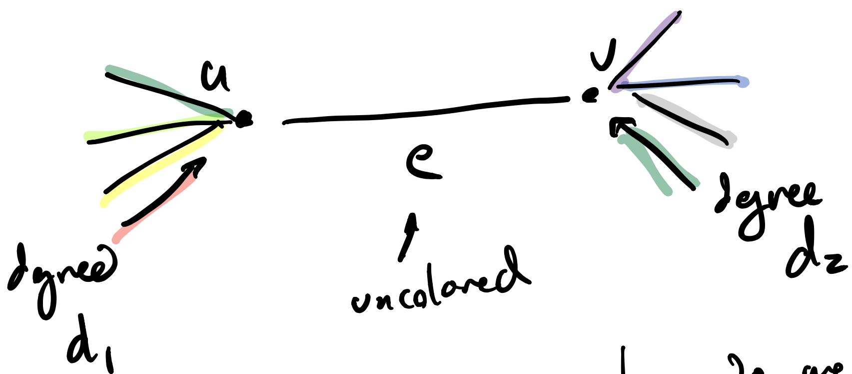
Vizing's theorem

$$\chi'(G) = \Delta(G) \text{ or } \Delta(G)+1$$

edge coloring: adjacent edges have different colors

$$\Delta(G) \leq \chi'(G)$$

Problem: Show $\chi'(G) \leq 2\Delta - 1$



Q1 how many colors do we need to make sure that we can choose a color for e, not yet used at u or v?

Q2 how many colored edges could be incident to u & v combined?

Why is it true that $\chi(G) \le \Delta + 1$?

- Proof by induction: 1

  induct on $\# V(G)$

  - if $\# V(G) = 1$ $\Delta = 0$ can color w/ 1 color ✓

  Base case.

  - if true for $\# V(G) < n$

    suppose $\# V(G) = n$.

    pick $v \in V(G)$, consider $G - v$. know $\Delta(G-v) \le \Delta(G)$

    so by induction can color w/ $\Delta(G-v)+1$ colors in $G-v$, so can def color w/ $\Delta(G)+1$ in $G-v$. Now can find some color of those $\Delta(G)+1$ not used in nbrs of v in G. use this to color v, done.

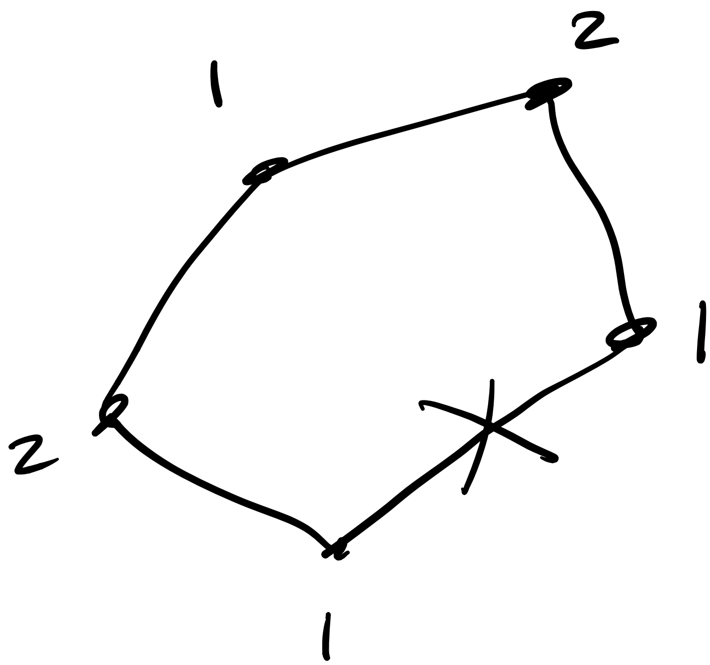
Alt. proof.

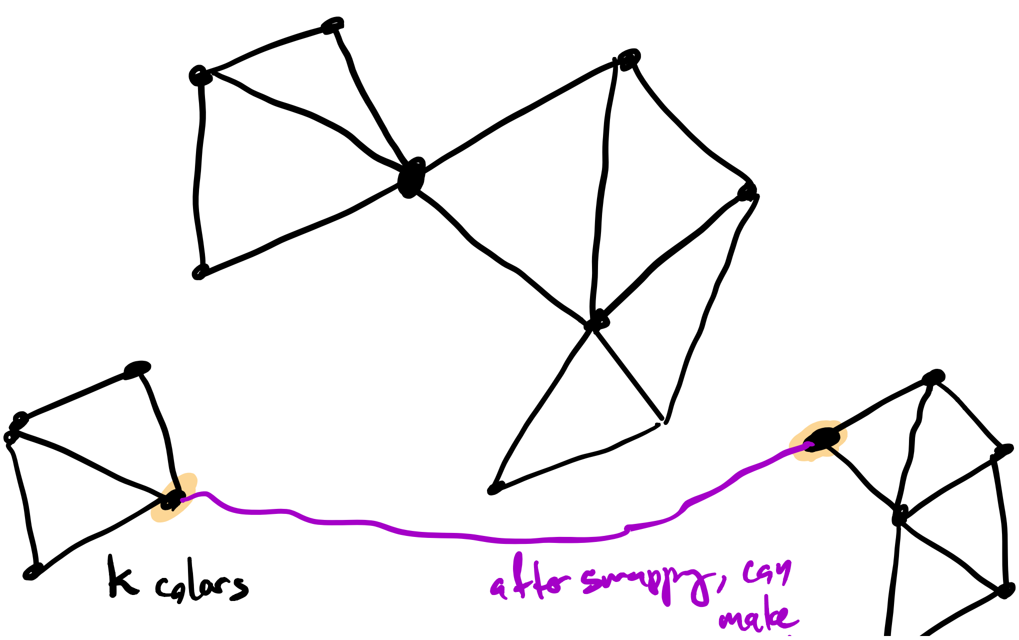
Fix a graph G, label the vertices $v_1, \dots, v_n$. Show can find colors for each v_i s.t. $v_1, \dots, v_i$ is valid coloring for each i. (when we get to $i = n$, done)

$i = 1$ choose any color.

assume chosen $v_1, \dots, v_{i-1}$

can find a new color for v_i not used by its neighbors as above

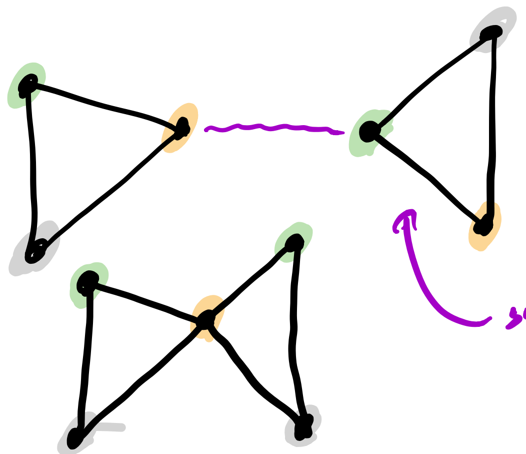

1
2
1
2
1


k colors
after swapping, can
make

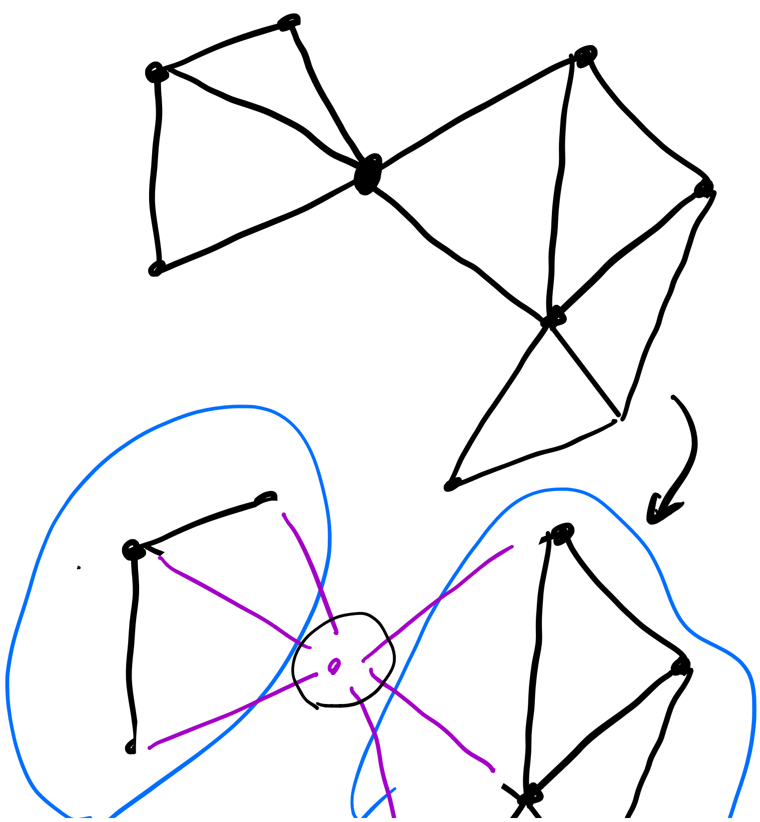
colors at attaching vertex agree.

k colors

k color for original?



switch colors, then attach.



$\Delta(G)$

not just $\Delta(G)+1$

(need to relabel)