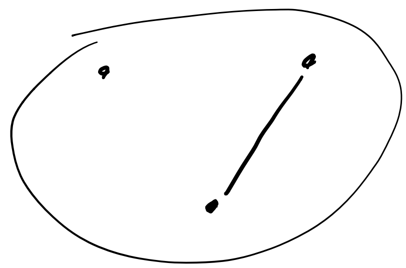
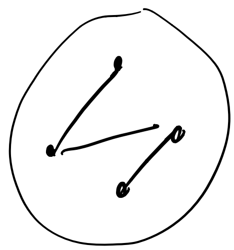
Definitions of a graph

Informally, a (simple) graph is a collection of vertices (dots) connected by edges





Def A (simple) graph consists of a set V of vertices and a relation called "incidence" / "adjacency"

Relation: $v \sim w$ means v connected to w by an edge

such that $v \sim w \iff w \sim v$, and $v \not\sim v$

ex: $V = \{1,2,3,4\}$

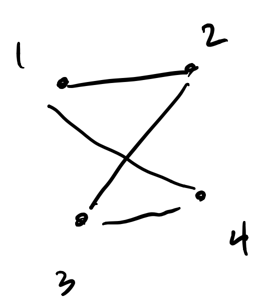
relation:

$1 \sim 2$ $\quad$ $2 \sim 1$

$2 \sim 3$ $\quad$ $3 \sim 2$

$3 \sim 4$ $\quad$ $4 \sim 3$

$4 \sim 1$ $\quad$ $1 \sim 4$



Altrnate def

A graph consists of two sets
V "vertices"
E "edges" together with an "incidence map"

$i : E \longrightarrow \mathcal{P}(V)$

$i(e)$ consists of a set of two $\overset{\text{distinct}}{\vee}$ vertices

i injective

prior example $V = \{1,2,3,4\}$
$E = \{a,b,c,d\} \left(= \{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\}\right)$

$i : E \longrightarrow \mathcal{P}(V)$
$i(a) = \{1,2\}$ $i(b) = \{2,3\}$ $i(c) = \{3,4\}$
$i(d) = \{1,4\}$

Def A graph is a matrix $(n \times n)$ such that
- only entries are 0 & 1
- symmetric - i.e. ij entry = ji entry
- 0 along diagonal

$i \sim j$ to mean 1 in ij slot

original graph:

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 1 & 0 & 1 & 0 \end{array}$$

Our working notation

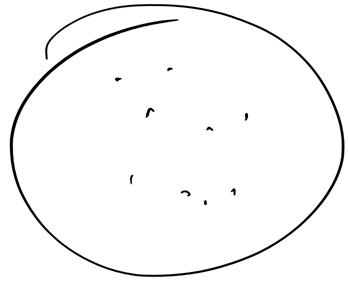
$G = (V, E)$ V vertices E edges

understood incidence relation describing which edge connects to which vertex.

examples

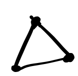
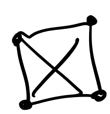
- trivial graph w/ n vertices

  $V = \{1, \dots, n\}$ $E = \emptyset$



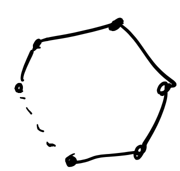
- complete graph w/ n vertices

  $V = \{1, \dots, n\}$ $E = \{\{ij\} \mid j \neq i\}$

  "K_n" $K_3 =$  $K_4 =$ 

- cycle graph w/ n vertices "C_n"

  $V = \{1, \dots, n\}$ $E = \{\{1,2\}, \{2,3\}, \{3,4\}, \dots, \{n,1\}\}$



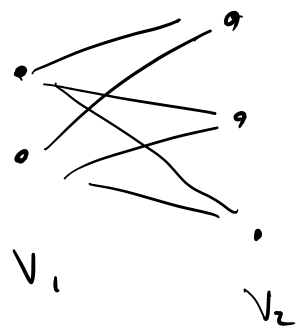
- Complete bipartite graph $K_{n,m}$

$V = V_1 \cup V_2$ disjoint union of sets V_1, V_2

V_1 = n elements V_2 = m elements

$E = \{\{v,w\} \mid v \in V_1, w \in V_2\}$

$K_{2,3}$



Basic Notions

Isomorphism

Given graphs $G_1 = (V_1, E_2)$ $G_2 = (V_2, E_2)$

we say G_1 is isomorphic to G_2 (write $G_1 \cong G_2$)

if we can find a map $\varphi: V_1 \longrightarrow V_2$ such that

$v, w \in V_1$ are adjacent if and only if $\varphi(v), \varphi(w) \in V_2$ are adjacent.

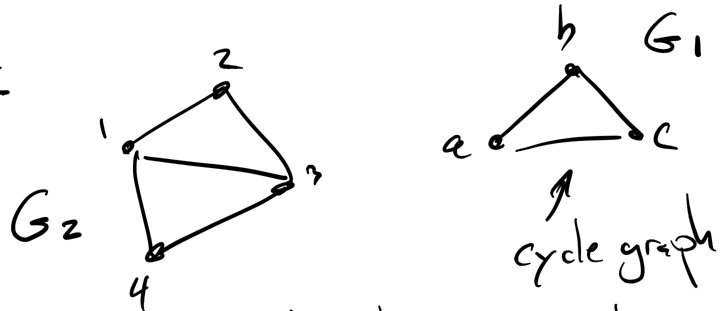
we say that φ is an <u>isomorphism</u> between G_1 & G_2.

Def Given graphs G_1, G_2, we say G_1 is a subgraph of G_2 if

$$G_1 = (V_1, E_1), \quad G_2 = (V_2, E_2)$$

$$V_1 \subset V_2 \text{ and } E_1 \subset E_2$$

Abuse of language



we'll often say G_1 is a subgraph of G_2 when we mean G_1 is isomorphic to a subgraph of G_2.

Conventional names C_3 "triangle" $= K_3$

C_4 = square

C_5 = pentagon

etc.

Deletions if G is a graph, $G = (V,E)$, $S \subset V$ some subset of vertices

we define $G - S$ to be graph $(V \setminus S, E')$

where E' are all edges in G not incident to any vertex in S.

Def if $T \subset E$, $G - T = (V, E \setminus T)$

__Def__ A spanning subgraph is a subgraph which contains all vertices of the original graph.

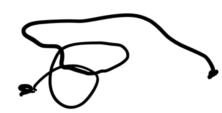
"__Def__" A Hamiltonian cycle is a spanning subgraph which is a cycle (spanning cycle)

__Hard question__: given a graph, is there a spanning (Hamiltonian) cycle.

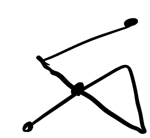
__Def__ A __walk__ in a graph $G = (V, E)$ is a sequence of vertices $v_1, v_2, \ldots, v_\ell$ such that v_i is adjacent to v_{i+1}, $i = 1, \ldots, \ell-1$.

"v_1-v_ℓ walk"

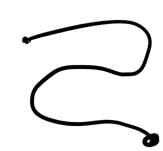
__Alternate def__ __walk__ = sequence $v_1 e_1 v_2 e_2 \cdots v_{\ell-1} e_{\ell-1} v_\ell$ where e_i incident to v_i & v_{i+1}



__Def__ A __trail__ is a walk with distinct edges



__Def__ A __path__ is a walk with distinct vertices



__Def__ A circuit is a trail which ends where it begins.

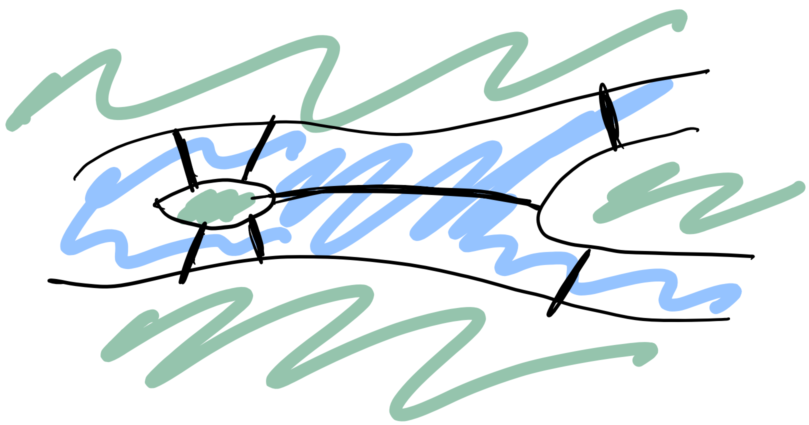
__Def__ A cycle is a walk, which ends where it begins, no repeated vertices except first & last.

Def A tour is a walk with every edge occuring at least once

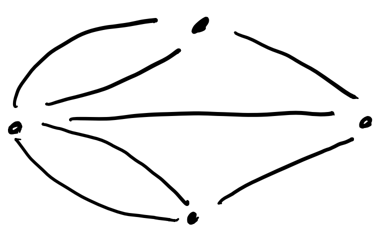
Def An Eulerian tour is a tour which is a trail

Def An Eulerian circuit is an Eulerian tour which is a circuit.

Königsberg Bridge Problem



Euler Circuit in



not quite a graph
- multigraph.

Definition a graph $G=(V,E)$ is connected if every pair of vertices is connected by a walk - i.e. $\forall v,w \in V$ $\exists$ a v-w walk in G.

<u>Define</u> v is connected to w in G if $\exists$ a v-w walk.

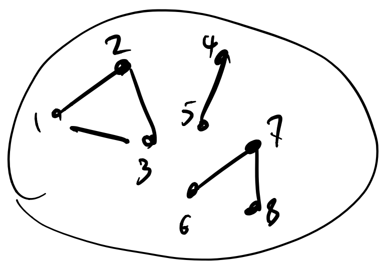
<u>Fact</u>: $G=(V,E)$, then we can uniquely decompose

$V = V_1 \cup \dots \cup V_e$ $V_i \cap V_j = \emptyset$ if $i \neq j$

such that v is connected to $w \iff v,w \in V_i$ some i.

V_i's "components" of G.

<u>Ex</u>:



$\{1,2,3\} \cup \{4,5\} \cup \{6,7,8\}$

"components"