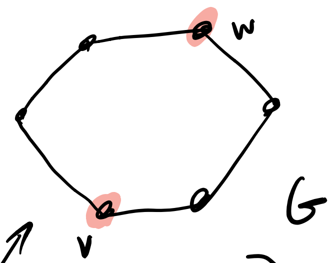
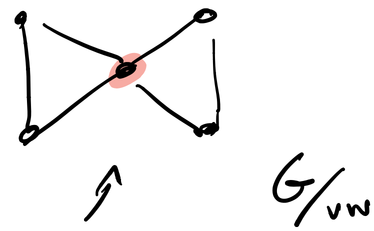
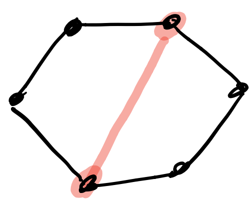




$\chi = 2$ $\qquad$ $\chi = 3$

is k colorable if and only if one of these is k-colorable.



$\chi = 2$

G + vw

$$\chi(G) = \min\{\chi(G/vw),\ \chi(G+vw)\}$$

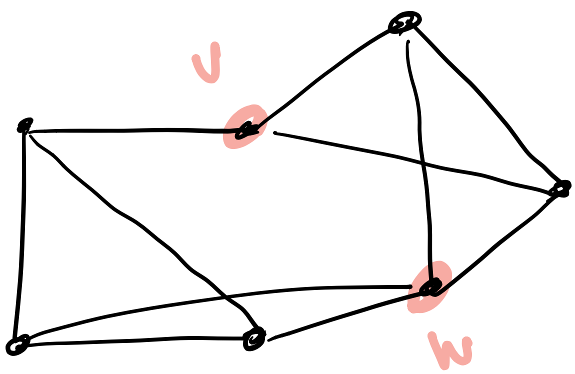
$\{$ways of k-colorg G$\}$ = $\{$ways of k-colorg G w/ v & w same color$\}$

$\cup$ $\{$ways of k-colorg G w v & w diffrent colors$\}$

bijection

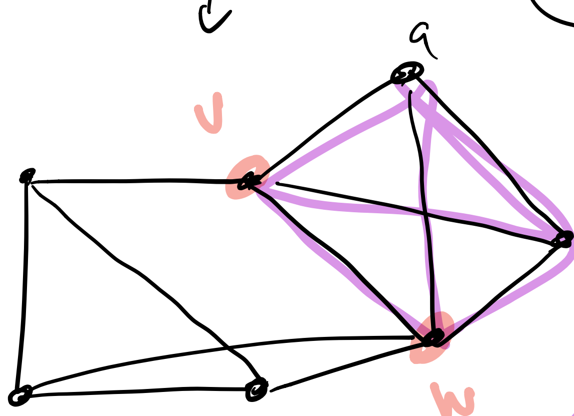
{ways of k-coloring G/vw}

{ways of k-coloring $G+vw$}

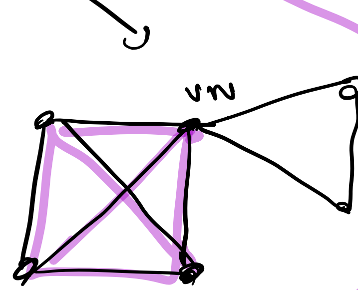


G

χ ? 4



$G+vw$ $\chi = 4$



$\chi = 4$

G/vw

Relating k-colorings of these 3 graphs

$$\begin{tikzcd}
 & G \arrow[dl] \arrow[dr, squiggly] & \\
G + vw & & G/vw
\end{tikzcd}$$

starting pt

$e = vw \qquad H = G + vw = G + e$

$$\begin{tikzcd}
 & H - e & H - e/vw \\
H \arrow[ur, squiggly, "\text{deletion}"] \arrow[rr, squiggly, "\text{contraction}"'] & & H/e = G/vw
\end{tikzcd}$$

New Perspective: How many k-colorings does H admit?

$\chi_G(k) = \#$ of k-colorings of G.

$\chi_G(k) > 0 \iff G$ is k-colorable

$\iff \chi(G) \le k$

$$\#\{\text{k-colorings of } G\} = \#\left\{\begin{matrix}\text{k colorings of G where}\\ \text{v \& w have same color}\end{matrix}\right\}$$

$$+ \#\left\{\begin{matrix}\text{k colorings of G where}\\ \text{v \& w have diff colors}\end{matrix}\right\}$$

$$= \#\{\text{k colorings of } G/vw\}$$

$$+ \#\{\text{k-colorings of } G+vw\}$$

$$\chi_G(k) = \chi_{G/vw}(k) + \chi_{G+vw}(k)$$

$$\chi_{H-e}(k) = \chi_{H/e}(k) + \chi_H(k)$$

$$\chi_H(k) = \chi_{H-e}(k) - \chi_{H/e}(k)$$

generally easier to use because one really "smaller" than H.

Main punchline: $\chi_H(k)$ is a polynomial function of k

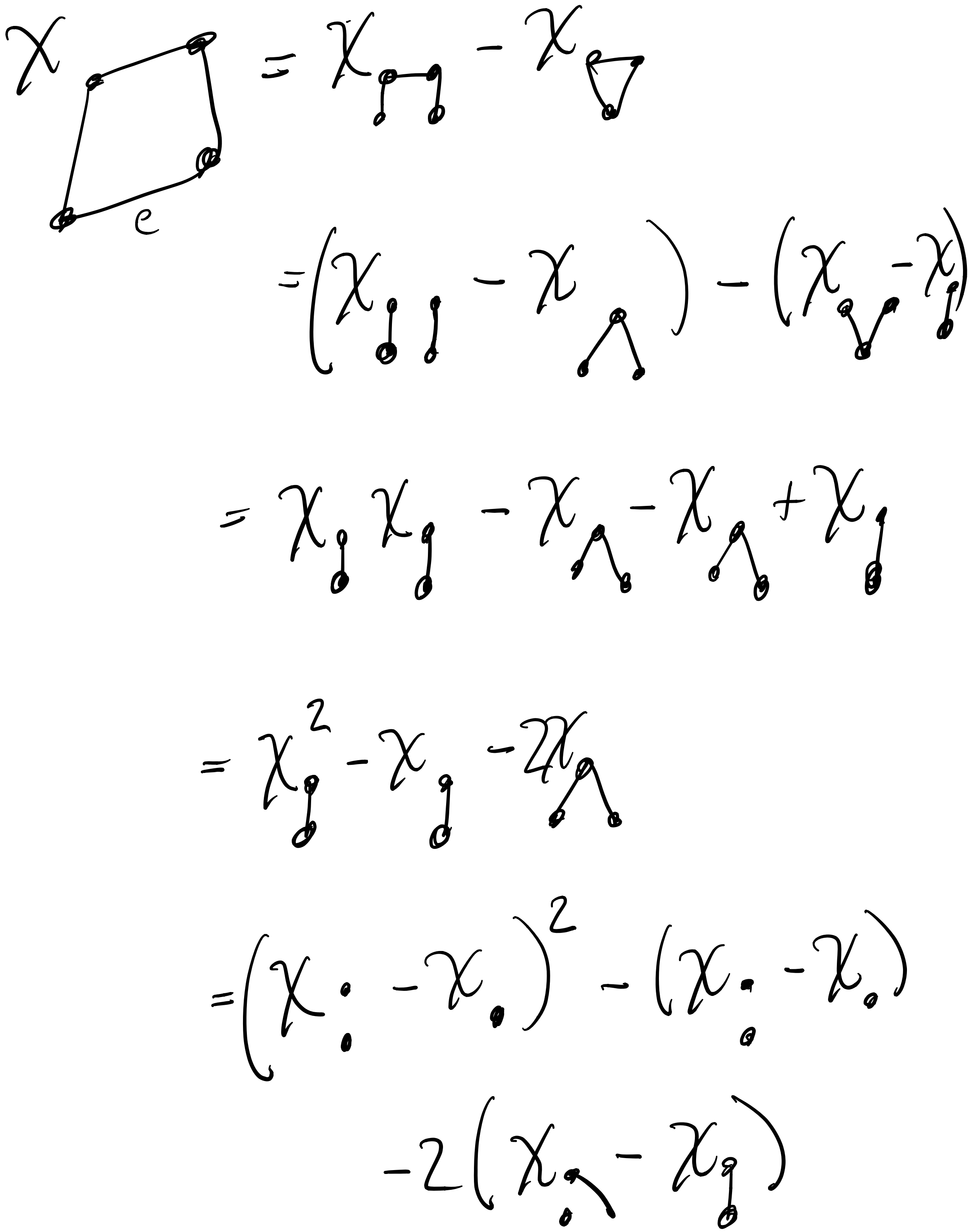
"Chromatic polynomial"

$$\boxed{\chi_{\bullet}(k) = k}$$

$$\chi_{\bullet\ \bullet}(k) = k^2$$

$$\chi_{\bullet\!-\!\bullet}(k) = \chi_{\bullet\ \bullet}(k) - \chi_{\bullet}(k)$$

$$= k^2 - k = k(k-1)$$


e

$$= (x_0^2 - x_0)^2 - (x_0^2 - x_0) - 2(x_0 x_{g_0} - x_{g_0})$$

$$= (x_0^2 - x_0)^2 - (x_0^2 - x_0) - 2(x_0 - 1)(x_{g_0})$$

$$x_{g_0} - x_0$$

$$= x_0^2 - x_0$$

$$= (k^2 - k)^2 - (k^2 - k) - 2(k-1)(k^2 - k)$$