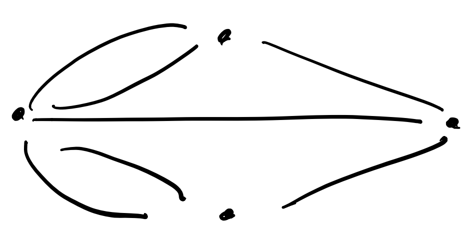
Recall:

Vocabulary:

- Walk
- Trail (no repeated edges)
- Path (no repeated vertices)
- Closed = end where it begins
- Circuit (closed trail)
- Cycle (closed walk w/ no repeated vertices except first & last)

- Eulerian tour (trail including all edges)
- Eulerian circuit (circuit including all edges)
- Hamiltonian path (path including all vertices)
- Hamiltonian cycle (cycle including all vertices)

Spanning subgraph = subgraph which includes all vertices.



Definition A multigraph (vertices & edges, multiple edges between (distinct) vertices allowed)

Pair (V,E) and a map $E \longrightarrow \mathcal{P}(V)$ ← subsets of vertices

edges

$e \longmapsto$ subset of exactly 2 vertices $\{v,w\}$

(not necessarily injective)

Matrix formulation

if we have n vertices, data of a multigraph is encoded by an $n \times n$ matrix (a_{ij}) $\quad a_{ij} =$ "# of edges between vertex i & j"

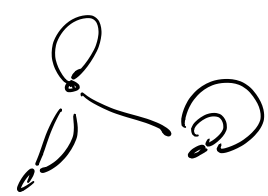
$a_{ij} \geq 0$ integers, $a_{ij} = a_{ji}$ $\qquad a_{ii} = 0$

Pseudograph: (loops are allowed)

(V,E) $\qquad E \longrightarrow \mathcal{P}(V)$

$e \longmapsto$ subset of order 1 or 2

no assumption of injectivity



Matrix formulation: (a_{ij}) $\quad a_{ij} \geq 0$ integer, $a_{ij} = a_{ji}$

Similar notions of sub multigraph sub pseudo graphs

spanning subpseudo graph, isomorphisms

(mapping of vertices of one graph to vertices of other &, edges to edges s.t. incident $\Leftrightarrow$ incident)

Some people

$$\text{others}\left\{\begin{array}{l}\text{Graph} = \text{pseudograph} \\ \text{loopless graph} = \text{multigraph} \\ \text{simple graph} = \text{graph}\end{array}\right\}\text{ us}$$

Euler solution to Königsberg bridge problem

Euler: A pseudograph has an Eulerian circuit if and only if it is connected, and every vertex has even degree

Def if $G = (V, E)$ is a pseudograph, $v \in V$,

$\deg v$ = # of edges incident to v (loops counted twice)

In fact also have

Euler: A pseudograph has an Eulerian tour which isn't a circuit if and only if it is connected and exactly 2 vertices have odd degree.

Note: if you know the statement for tours $\Rightarrow$ know it for circuits & vise-versa.

Clear part: if have an Eulerian circuit $\Rightarrow$ all vertices have even degree

(have to exit & enter each vertex same # of times)

if an Eulerian tour $\Rightarrow$ exactly 2 odd

(source exits once more than enters, terminus enters one more than exists)

sorry bad grammar.

(if exactly 2 odd $\Rightarrow$ Euler tour)

$\Downarrow$?

(all even $\Rightarrow$ Euler circuit)

G has all vertices even

pick an edge e and remove it
say it connected v, w
together.

in new graph G-e
v & w have odd degree
⇒ can find a tour from
v to w. continue this
through e to get back
from w to v
⇓
Euler circuit in G.

what if e is a loop?
A: pick one that's not a loop.
maybe there are only loops!
⇒ only one vertex = v
then the Eulerian circuit
just successively goes through
each loop at v, these are all edges, done.

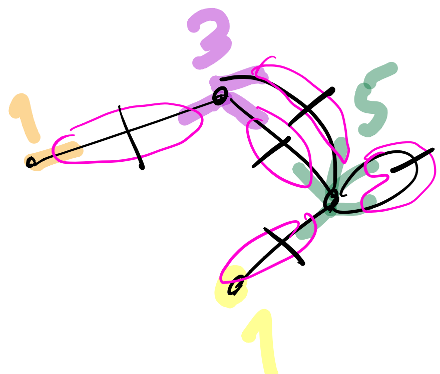
what if G-e is not
connected?

(can't happen)

"Degree formula"

<u>Theorem</u>: if G is a pseudograph then

$$\sum_{v \in V} \deg v = 2\#E$$



split each edge into 2 "half edges"

each ½ edge belongs to exactly one vertex and $\deg v = \#$ of half edges at v

$$\sum \deg v = \sum \tfrac{1}{2}\text{edges} = 2\#E.$$

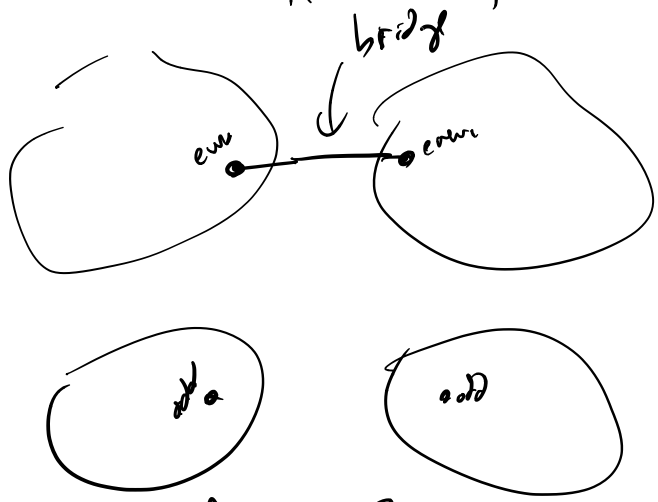
<u>Cor</u> if G is any graph, there must always be an even # of vertices of odd degree.

Can't have a graph with exactly one odd vertex.

<u>Definition</u> An edge e in a connected graph G is called a <u>bridge</u> if $G-e$ is not connected.

<u>Prop</u> if G is connected, all vertices even degree $\Longrightarrow$ G has no bridges.

if bridge ... problem:



each is now a graph w/ exactly one vertex of odd degree

$\Rightarrow$ impossible!

so no bridges can exist