Last time:

towards Eulerian circuits & tours

Goal:

Theorem 1: If a pseudograph is connected & all its vertices have even degree, then it has an Eulerian circuit

Strategy for proof:

Step 1: prove a related theorem about Eulerian tours

Thm 2: If a pseudograph has exactly two vertices v, w of odd degree and is connected, then there is an Eulerian tour from v to w.

Step 2: Notice that if theorem 2 is true then theorem 1 is also true.

Last time

Now: proof of theorem 2

Suppose we have a pseudograph, which is connected & all vertices have even degree except v, w.

Strategy:

Pick $x \neq v, w$ connected to v, start w/ edge e from v to x, now by induction, can find an Eulerian tour from x to w in $G-e$

in this graph, x has odd degree (even $-e$)

all other vertices have same degree (don't involve e)

v has even degree (odd $-e$).

add e to begining of the tour from x to w in $G-e$ and get a tour from v to w which is Eulerian.

why is $G-e$ connected?

If $G-e$ is not connected then its because can't get from v to x

FALSE

~~Claim: deg $v = 1$~~ ~~if this was true, could consider $G-v$ instead of $G-e$~~

~~⇑ is connected~~

maybe its not

Instead two possibilities:

if $G-e$ is still connected, use induction

if $G-e$ is disconnected, have two components, one with x & one with v

since odd number of odd deg vertices, w must be in this component

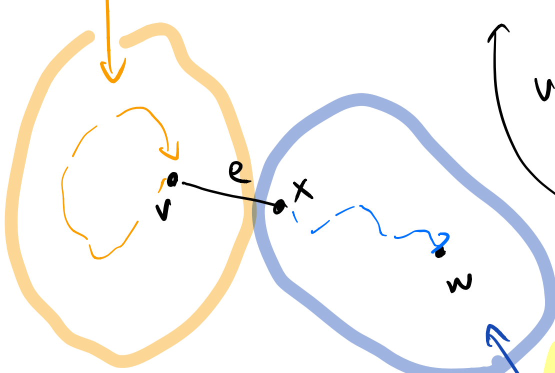
all even degrees.

by thm 2 on this smaller graph, can find an Euler tour from x to w.

by thm 1 on this smaller graph can find a Euler circuit.

$v \xrightarrow{\text{via circuit}} v \xrightarrow{e} x \xrightarrow{\text{via tour}} w$

have an Euler circuit by thm 1



has an Euler tour by thm 2

Thm 2 for at most n edges $\Rightarrow$ Thm 1 for at most $n+1$ edges

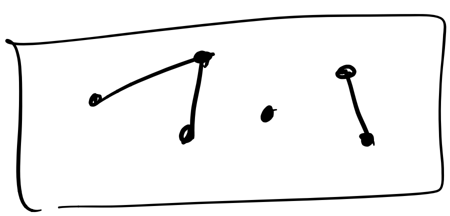
Induction to prove thm 2

Assume Thm 2 true for up to n edges.

To show Thm 2 for $n+1$ edges, reduce to theorem 2 for at most n edges & theorem 1 for at most n edges

---

"Recall" If G is a graph, we define $\omega(G) = \#$ of connected components in G



G

$\omega(G) = 3$

Useful fact: If $H \subset G$ is a spanning subgraph (all vertices) then $\omega(H) \geq \omega(G)$

Another useful fact: if $G = C_n$ cyclegraph w/ n vertices, S a collection of k edges in G $(k \leq n)$, $(k \geq 1)$ then / T a collection of k vertices

$\omega(G-S) = k$ / $\omega(G-T) \leq k$

Recall G is Hamiltonian if G has a spanning subgraph which is a cycle.

Shortest path problem:

Given a (simple) graph G, together with "weights" assigned to edges

$\omega : E \longrightarrow \mathbb{R}_{\geq 0}$

given a starting & ending vertex v, w

Goal: Efficiently find a path from v to w

$v e_1 x_1 e_2 x_2 \cdots e_{n-1} x_{n-1} e_n w$ such that

$\sum_{i=1}^{n} \omega(e_i)$ is minimal

Amazingly, this can be done: Dijkstra's algorithm.