Level 2 introduction to graph theory

Notation: if S is a set, $\mathcal{P}(S)$ = the power set of S
= set of subsets of S

$$\#S = n \qquad \#\mathcal{P}(S) = 2^n$$

$$\mathcal{P}_k(S) = \{T \in \mathcal{P}(S) \mid \#T = k\}$$

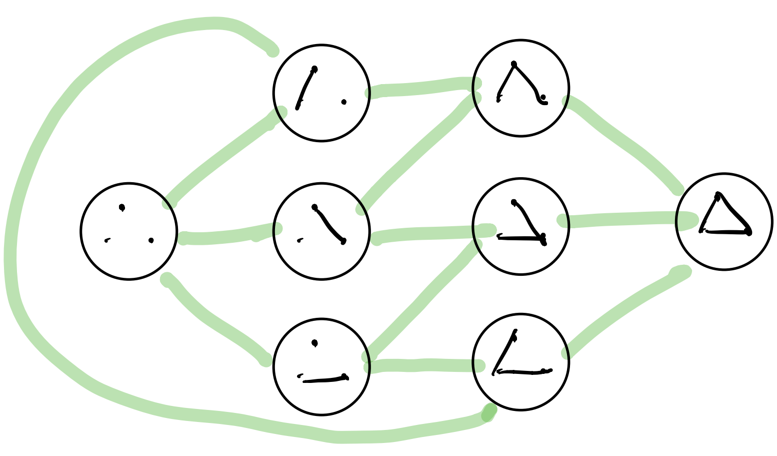
$$\#\mathcal{P}_k(S) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Def A (simple) graph is a pair $G = (V, E)$ where V is a nonempty set, and E is a subset of $\mathcal{P}_2(V)$

Unless we say otherwise, we will assume that V is finite.

Ex: consider $V = \{\text{graphs with vertex set } \{1,2,3\}\}$

$$E = \{\{G_1, G_2\} \mid G_1 \text{ \& } G_2 \text{ differ by at most one edge}\}$$



Given a graph $G=(V,E)$, we will write $V(G)$ for V & $E(G)$ for E

Def A subgraph H of a graph G is a graph such that $V(H) \subset V(G)$ & $E(H) \subset E(G)$. (Notation $H \subset G$)

Def A subgraph $\underset{\vee}{H \subset G}$ is is proper if either $V(G) \neq V(H)$ or $E(G) \neq E(H)$

Def A subgraph $H \subset G$ is spanning if $V(H) = V(G)$.

If G is a graph, $S \subset V(G)$, we define

$G[S]$ = subgraph w/ $V(G[S]) = S$

"vertex induced subgraph"

$$E(G[S]) = \{e \in E(G) \mid e \in P_2(S)\} = E(G) \cap P_2(S).$$

If $X \subset E$, define $G[X]$ = subgraph induced by edges X

$E(G[X]) = X$ $\quad$ $V(G[X])$ = all vertices incident to edges in X

$$= \bigcup_{e \in X} e$$

Def a subgraph $H \subset G$ is called a component of G if H is connected, and if $H \subset H' \subset G$, $H \neq H'$ then H' is not connected.

= "H is a maximal connected subgraph"

Observation: if H is a component of G and $v \in V(H)$

let $S = \{ w \in G \mid \exists \text{ a walk from } v \text{ to } w \}$

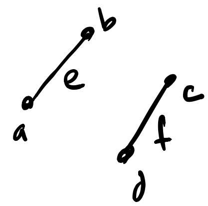
then $H = G[S]$

Alt defs

$$G - v = G[V(G) \setminus \{v\}]$$

$$G - e \neq G[E(G) \setminus \{e\}]$$ (sometimes same but not always)

this never removes vertices



this sometimes removes vertices

$G - e$ $\quad V(G-e) = V(G)$

$E(G-e) = E(G) \setminus \{e\}$

Connectedness & components

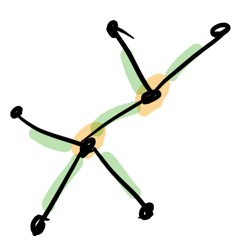
$V = \{1, 2, 3, \ldots, 10\}$

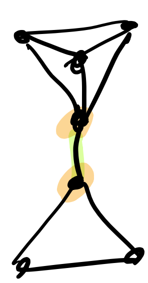
$E = \{12, 14, 25, 37, 39, 4(10), 32, 56, 59, 62, 71, 42\}$

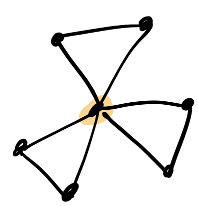
	1	2	3	4	5	6	7	8	9	10
1	0	0	0	0	1	1	1	0	1	1
2	0	0	1	1	0	1	0	0	1	0
3										
⋮										

Def $k(G) = \#$ of components

Q if G is connected, how can we quantify how connected it is?







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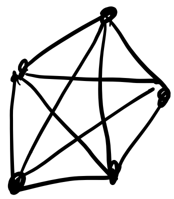

Def If G is connected, $e \in E(G)$, we say e is a bridge if $G-e$ is disconnected

More generally, if G is not necessarily connected, we say $e \in E(G)$ is a bridge if it is bridge in one of the compnts of G.

Alternately: $e \in E(G)$ is a bridge if $k(G) < k(G-e)$

Def: $v \in V(G)$ is a **cut vertex** if $k(G) < k(G-v)$

Q: if e is a bridge in a connected graph G, $e = \{v,w\}$ then either $G =$  or either v or w is a bridge?

Yes: if say v connected to $u \neq w$ and $G-v$ is connected, then we have a path from u to w in $G-v$, but then would have a cycle $u \rightsquigarrow w \overset{e}{—} v$. but if e is part of a cycle, it can't be a bridge.

Theorem 5.1 if a vertex v is incident to a bridge e then it is a cut vertex if and only if $\deg v \geq 2$.

Theorem 4.1 an edge $e \in E(G)$ is a bridge if and only if it is not on a cycle.

Theorem Every graph must have at least two vertices which are not cut vertices.

Def if $v, w \in V(G)$ and P is a path from v to w such that length of P is as small as possible, we say say P is a geodesic, and we define $\partial(v,w) =$ length of P.

Theorem: if G is a connected graph, $v \in V(G)$, $u \in V(G)$ which is as far as possible from v, then u is not a cut vertex.