Given a graph G, $k > 0$

Is G k-colorable? i.e. is $\chi(G) \leq k$?

<u>Approach:</u> Reduce to simpler graphs

But: what makes a graph simpler?

For this algorithm:

- More edges (!)
- Fewer vertices

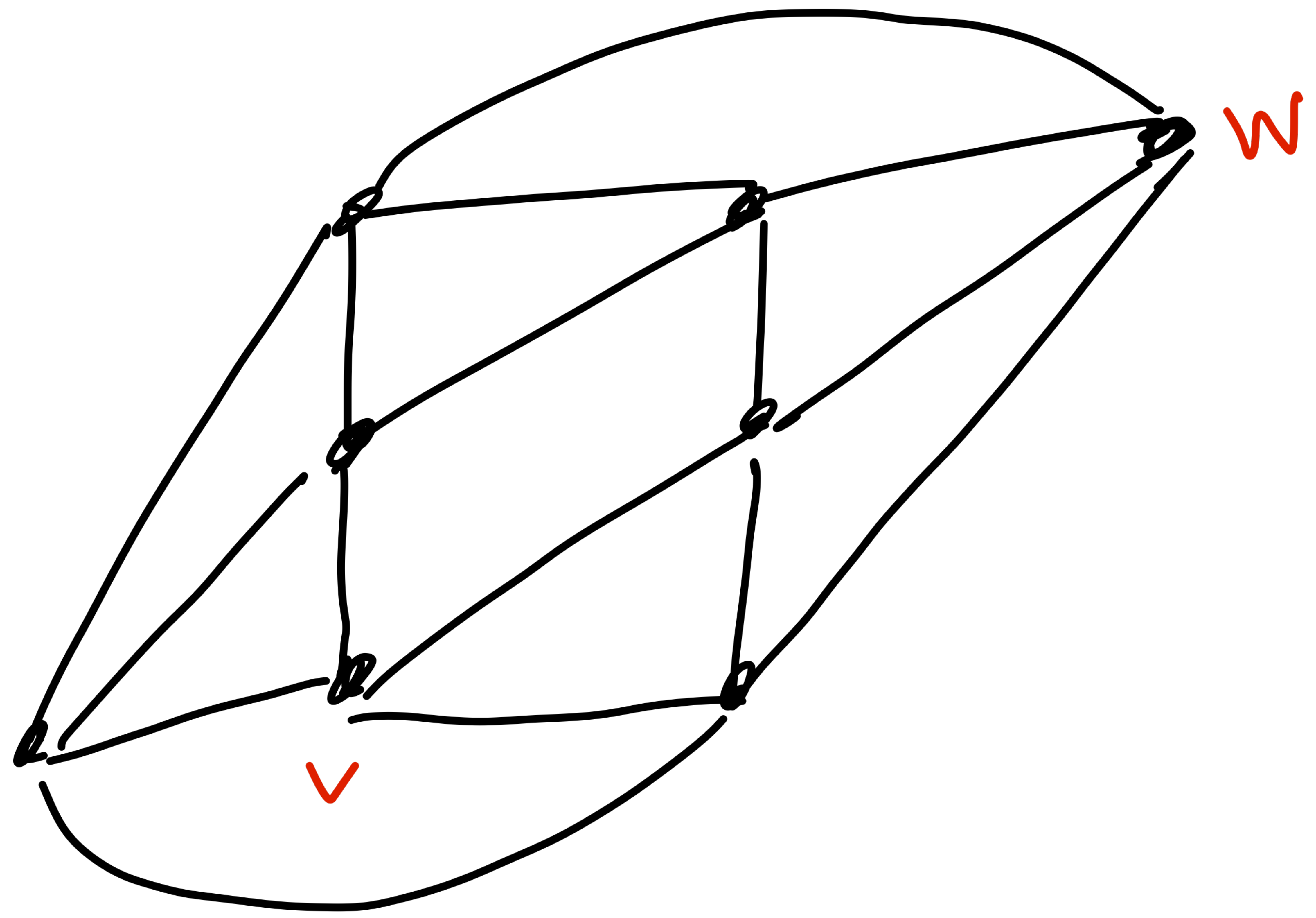
each makes $\chi(G)$ easier to compute!

Idea: Choose two vertices in the graph we are trying to color

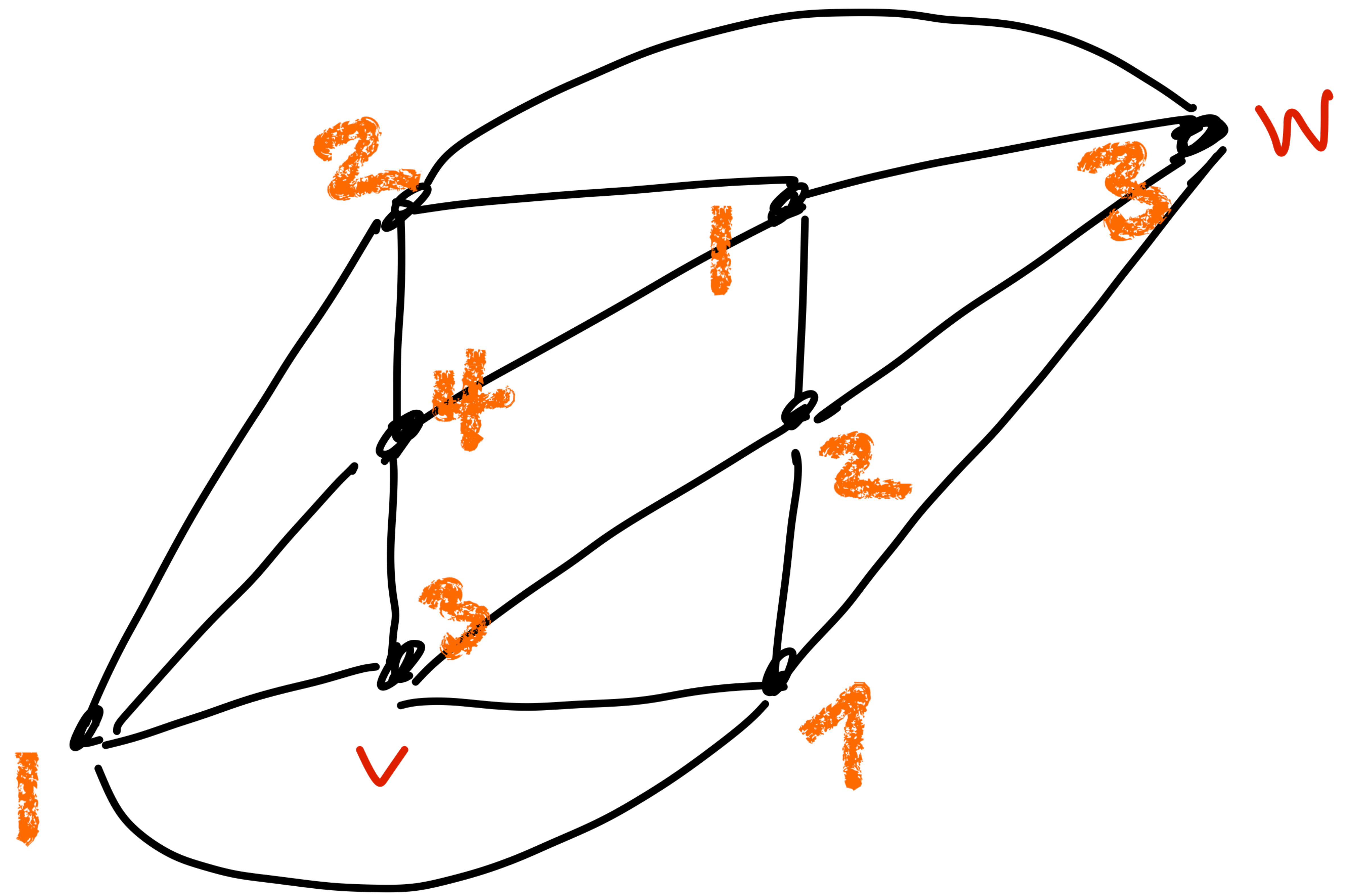
either v & w can be assigned same color or different colors

coloring so that they have different colors $\longleftrightarrow$ coloring w/ edge added

coloring so they have same color $\longleftrightarrow$ coloring w/ v & w identified as same vertex



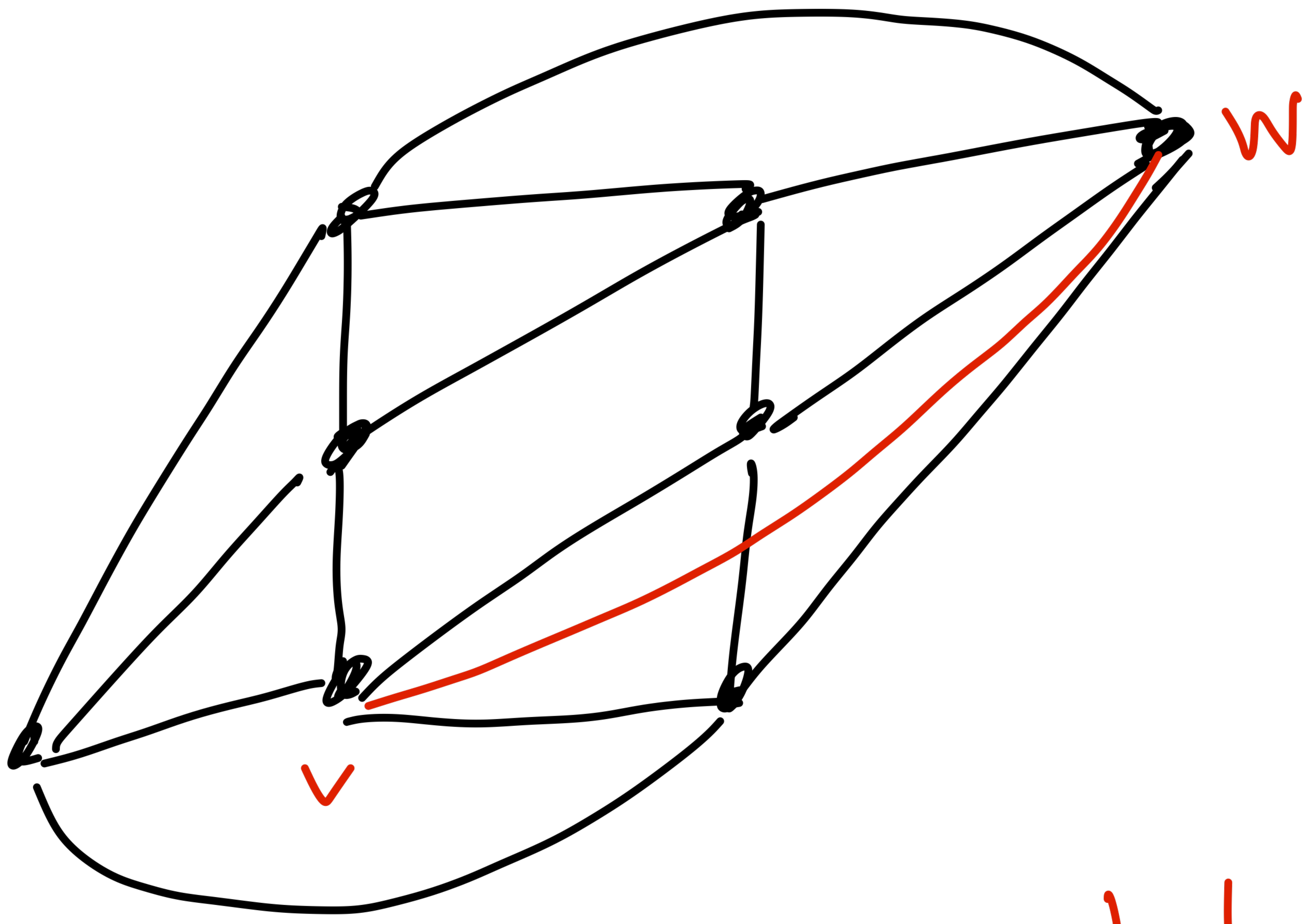
How many colors?



How many colors?

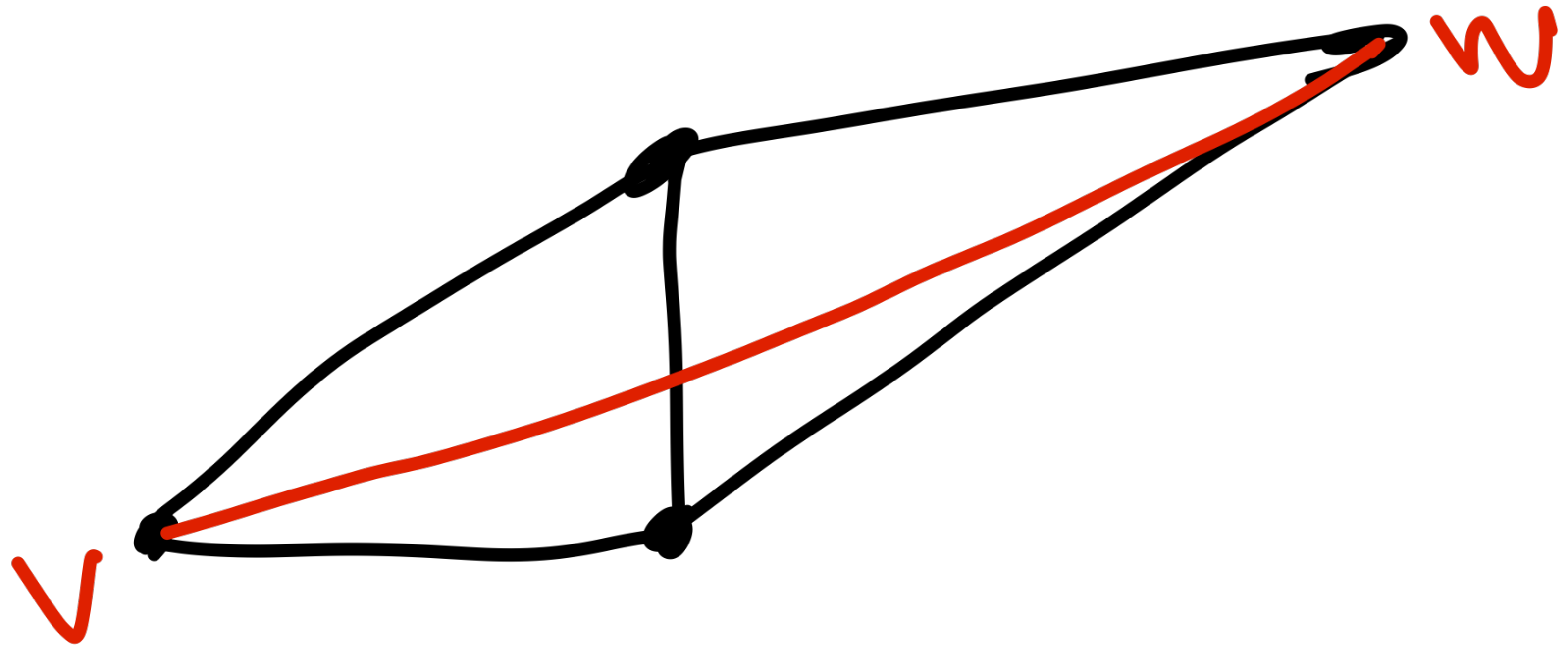
4?!

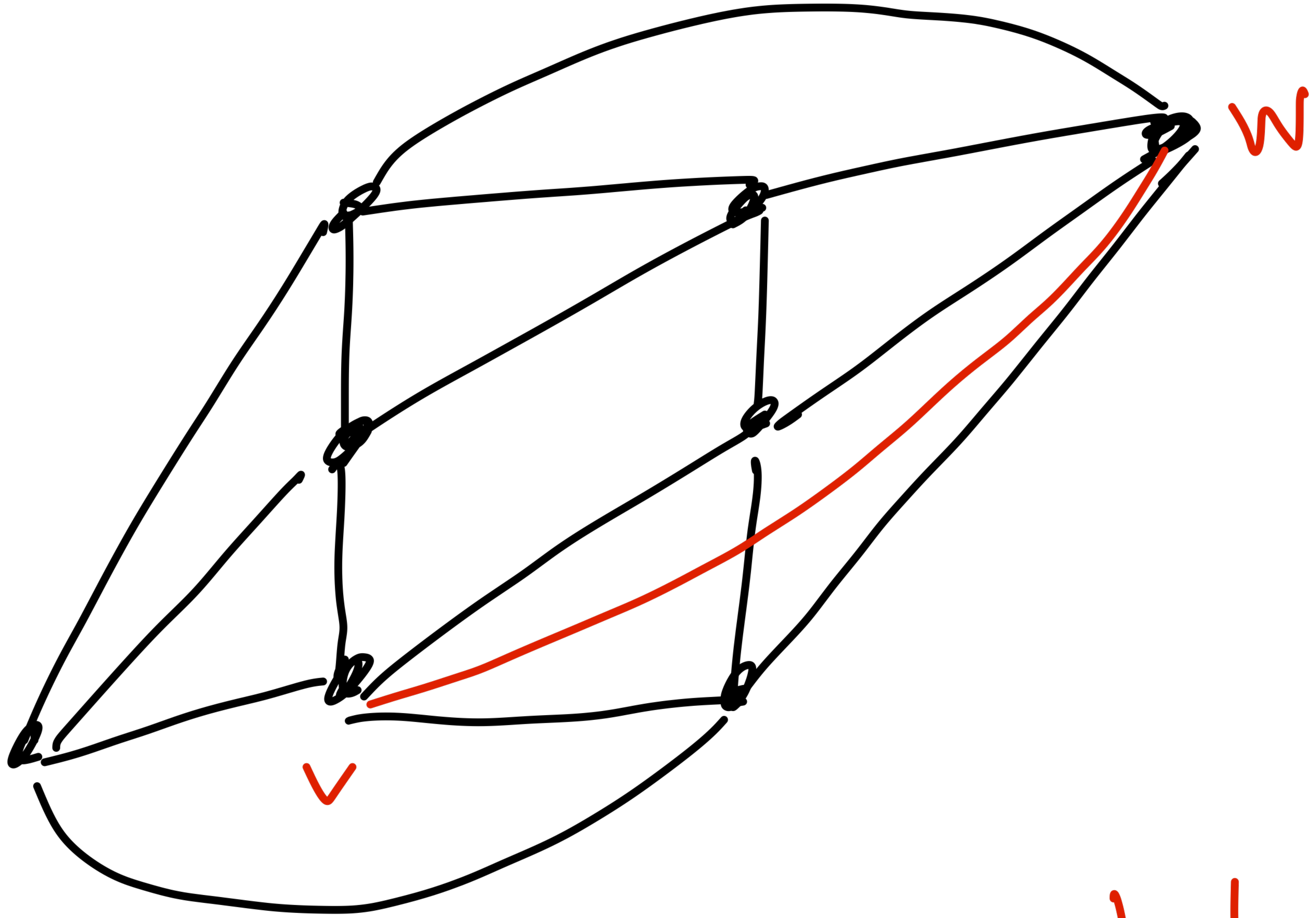
why not 3?



coloring v & w w/ different colors is equivalent to coloring this graph

but this has a K_4!





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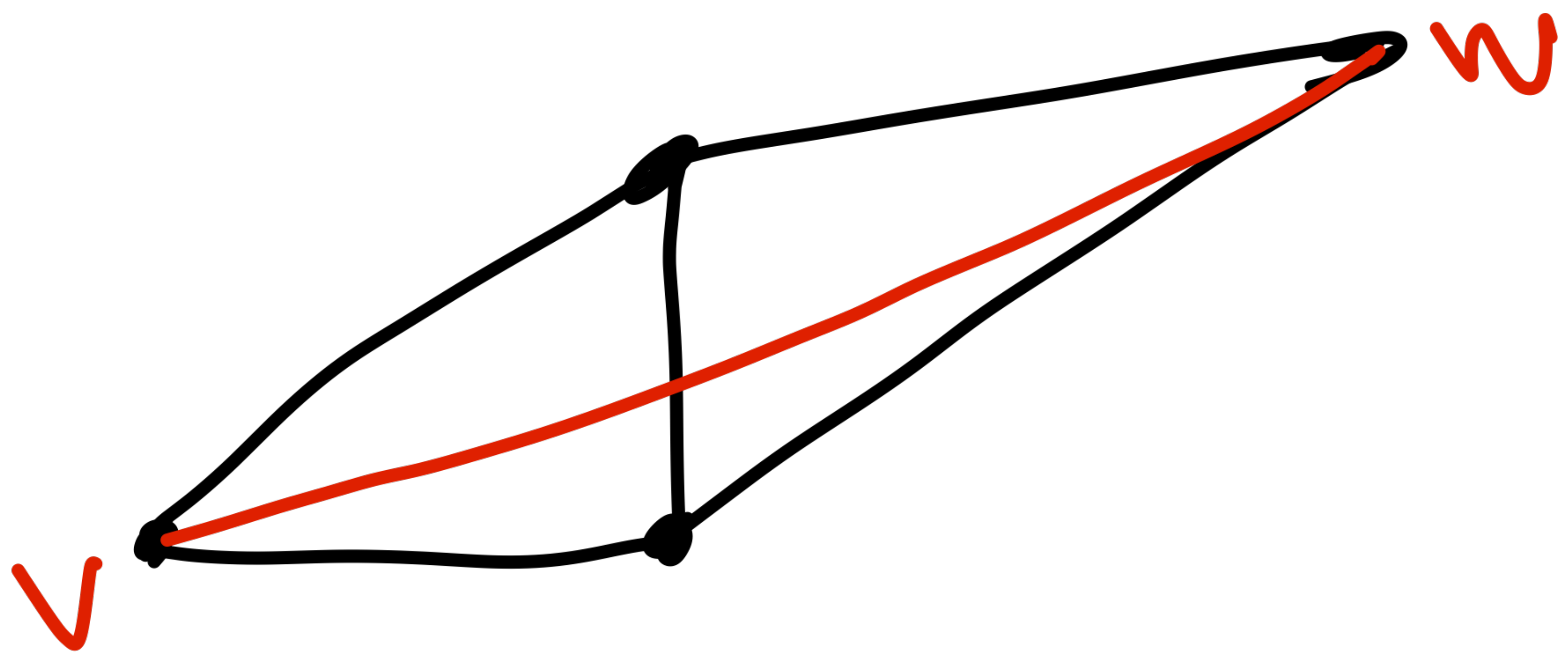
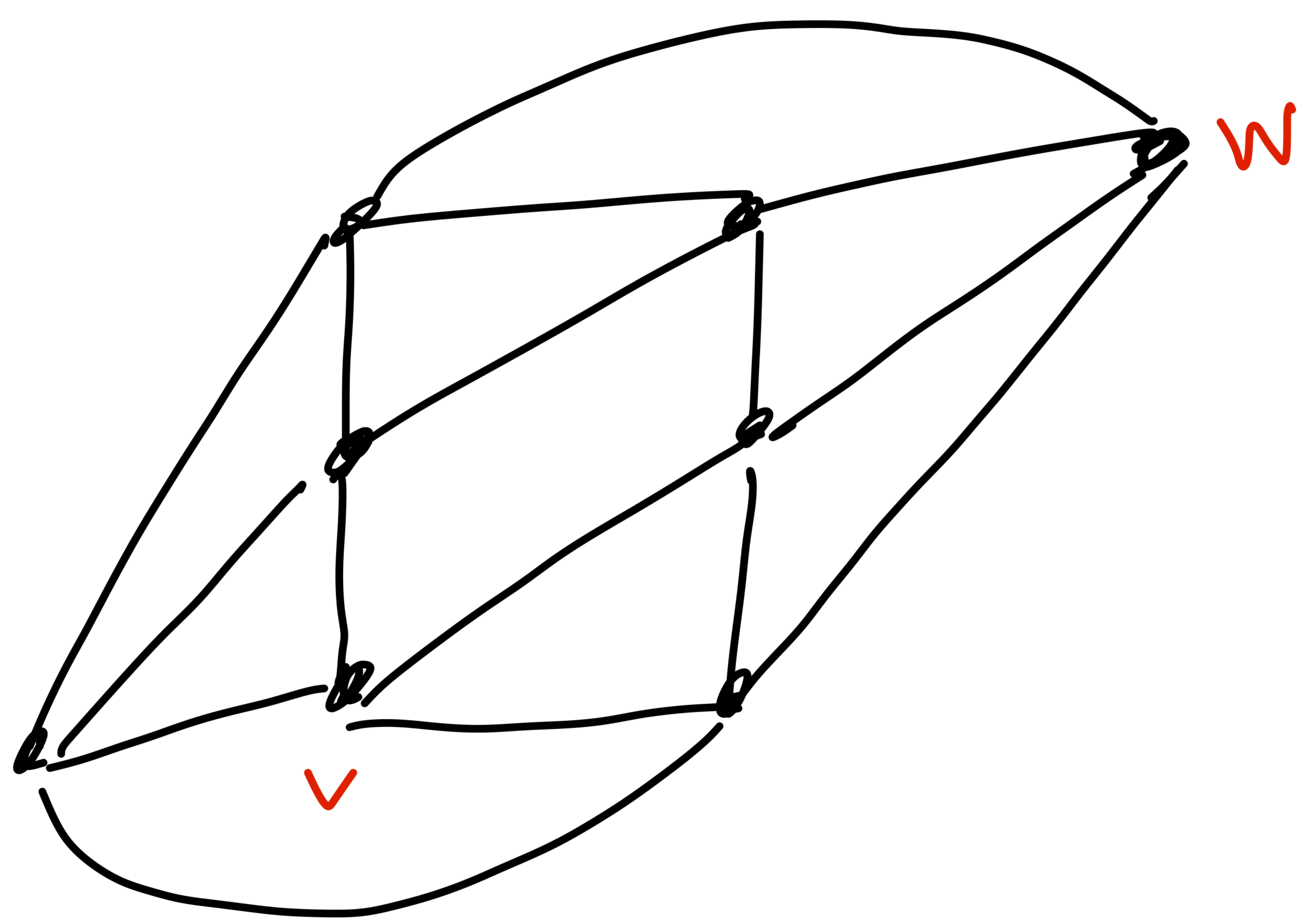
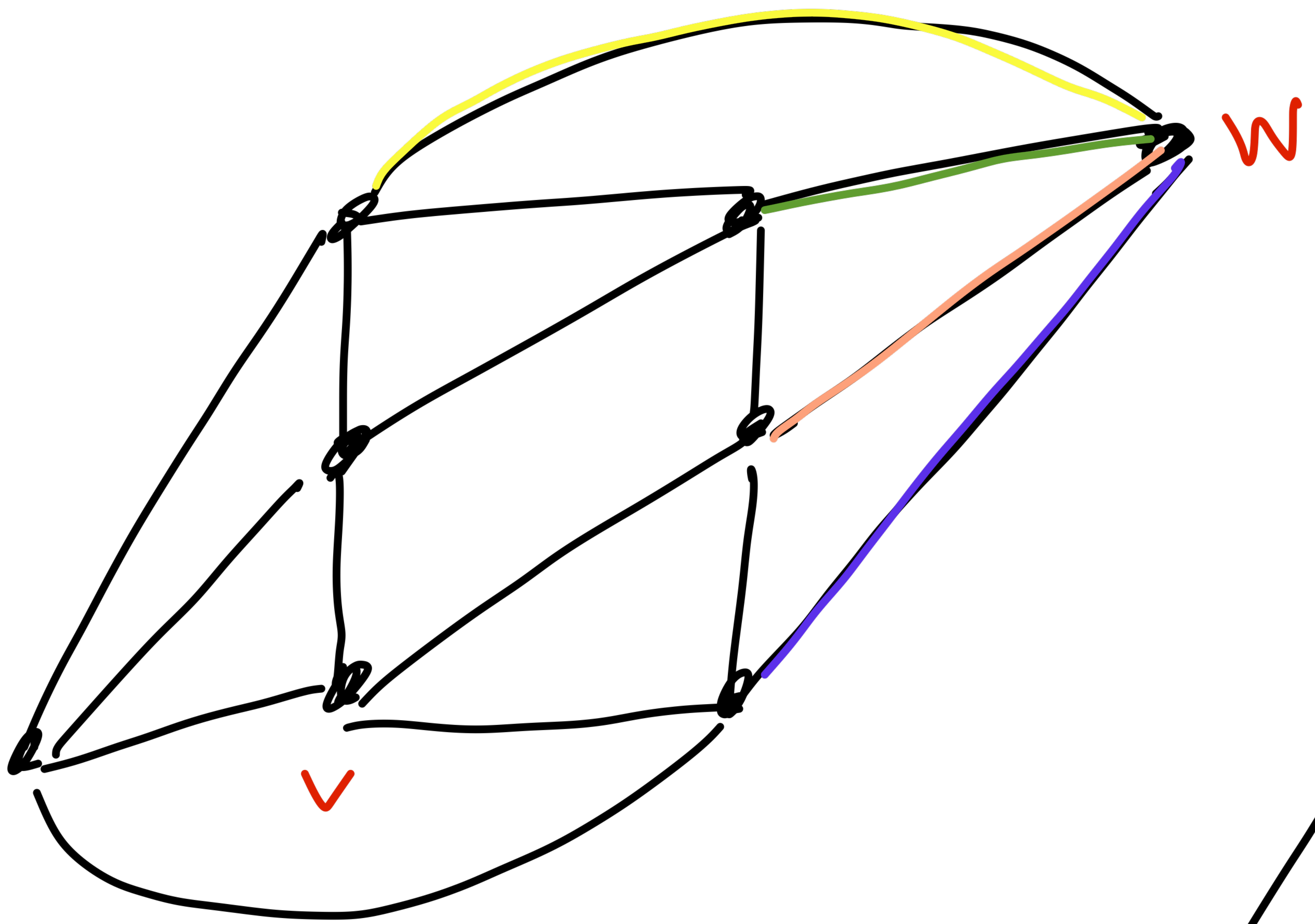


conclusion:

if v & w have different colors ⟹ need 4 colors!



what if v & w have the same color?



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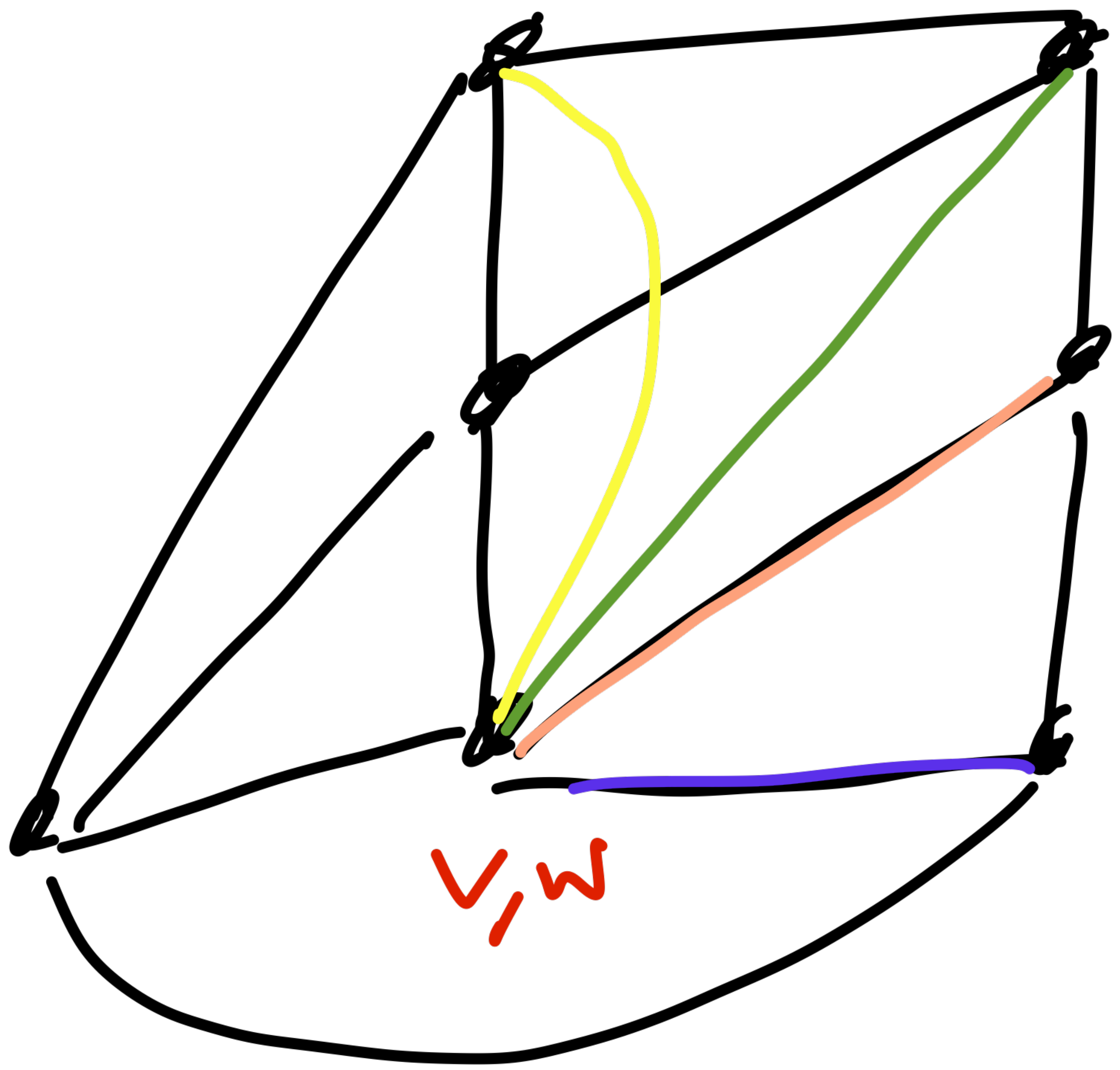
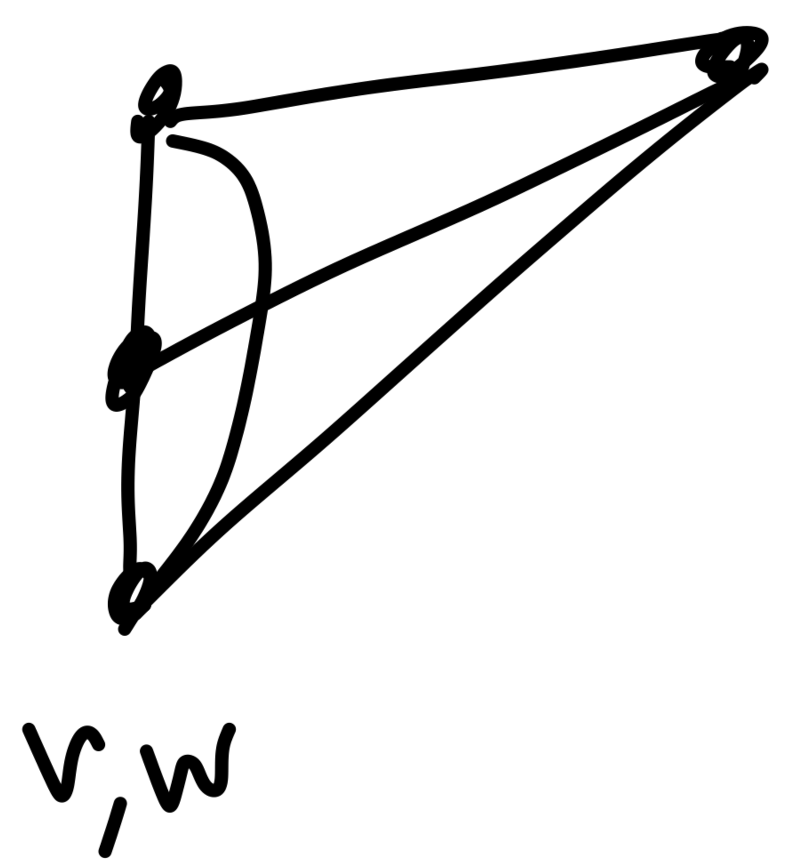
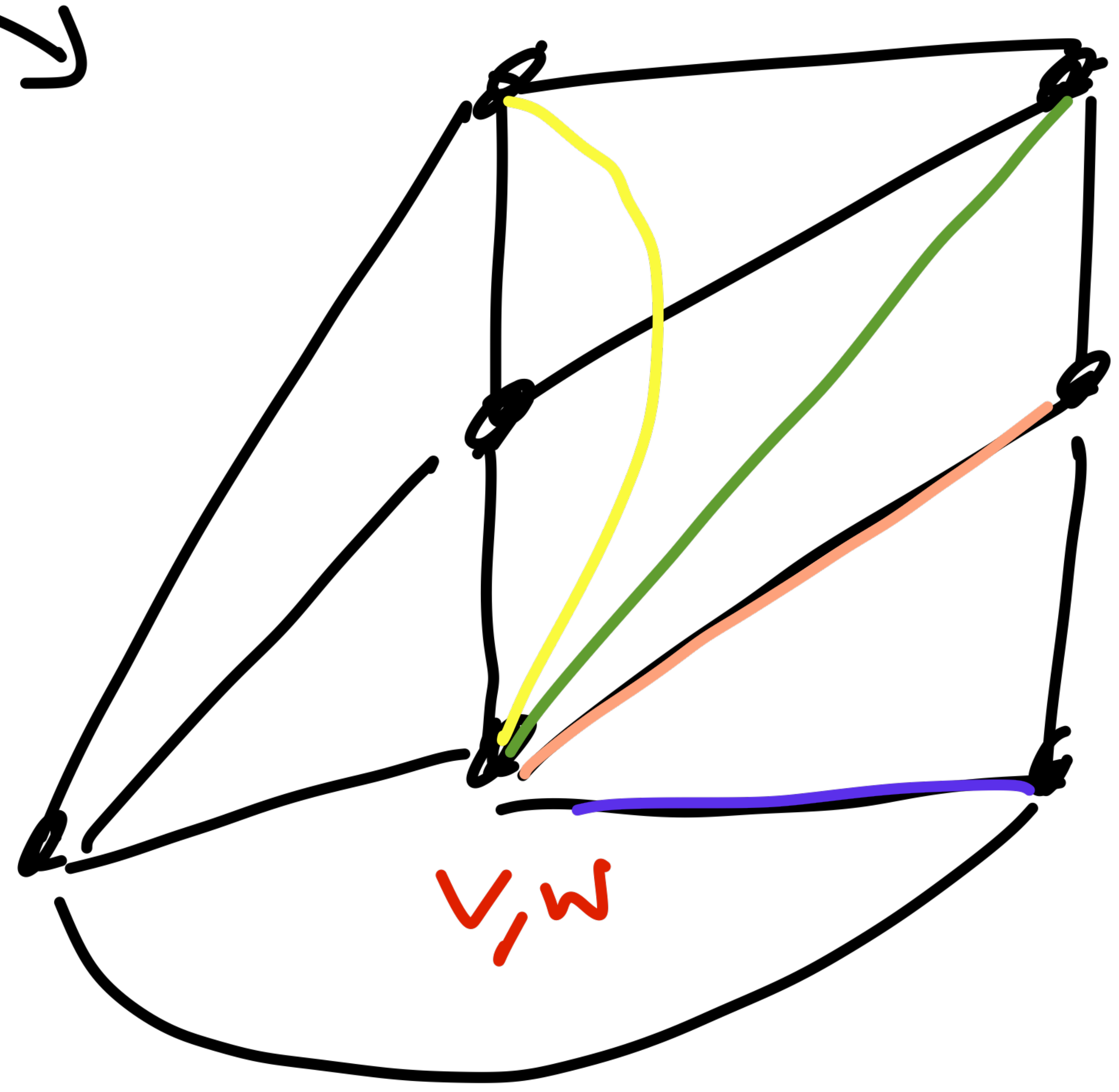


Same as coloring this graph w/ v & w identified new graph has a K_4!





so need at least 4 colors if v, w are given the same color!

<u>Proposition</u> Let G be a graph, $v, w \in V(G)$.
Then there is a bijection between the following sets:

$$\left\{ \begin{array}{c} k\text{ colorings of } G \\ \text{in which } v \,\&\, w \text{ are assigned} \\ \text{the same color} \end{array} \right\} \qquad \left\{ \begin{array}{c} k\text{ colorings} \\ \text{of } G/vw \end{array} \right\}$$

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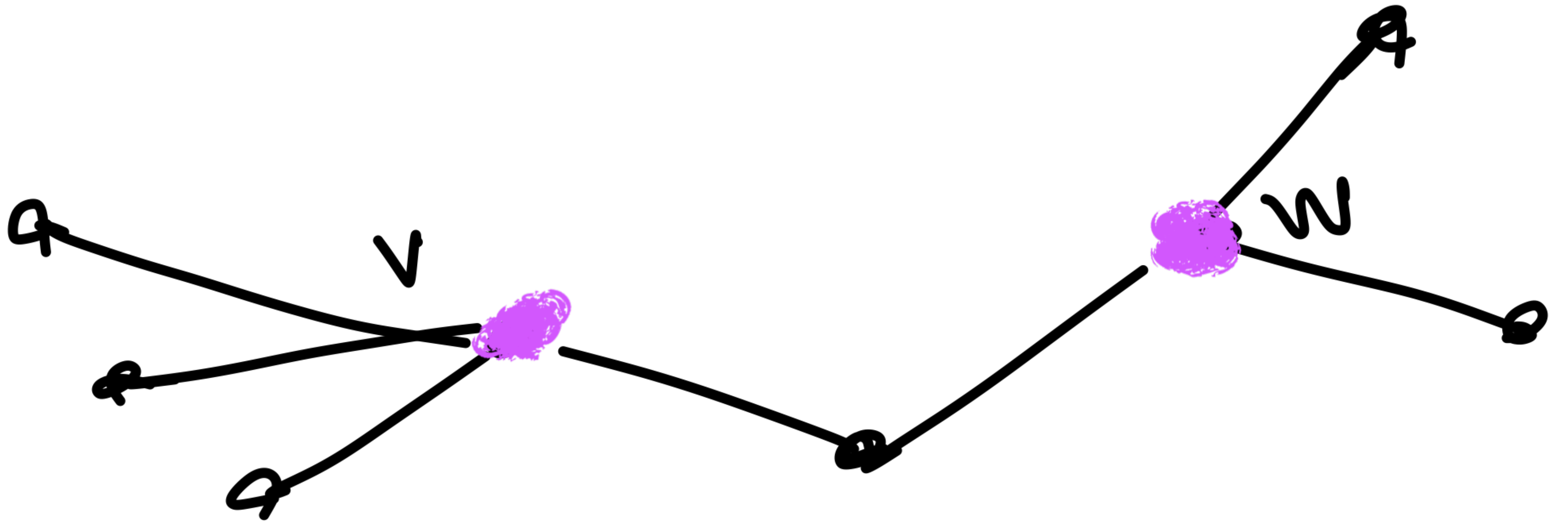
Suppose we have a k-coloring of G which assigns v & w the same color



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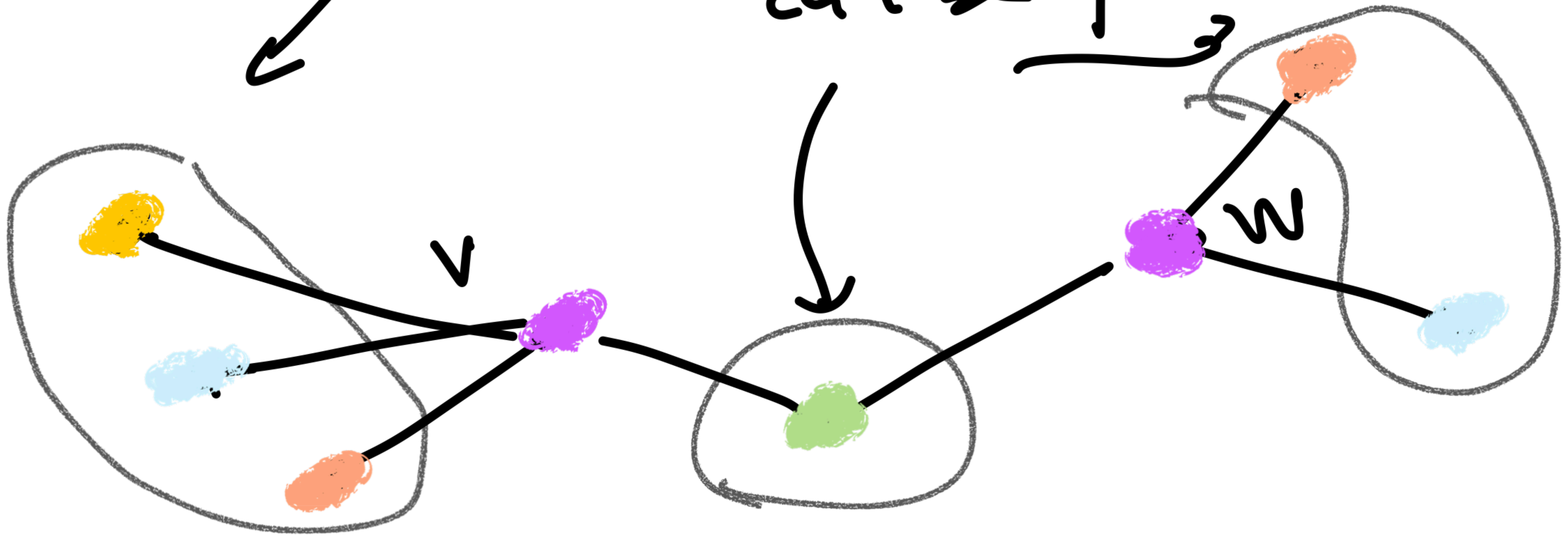

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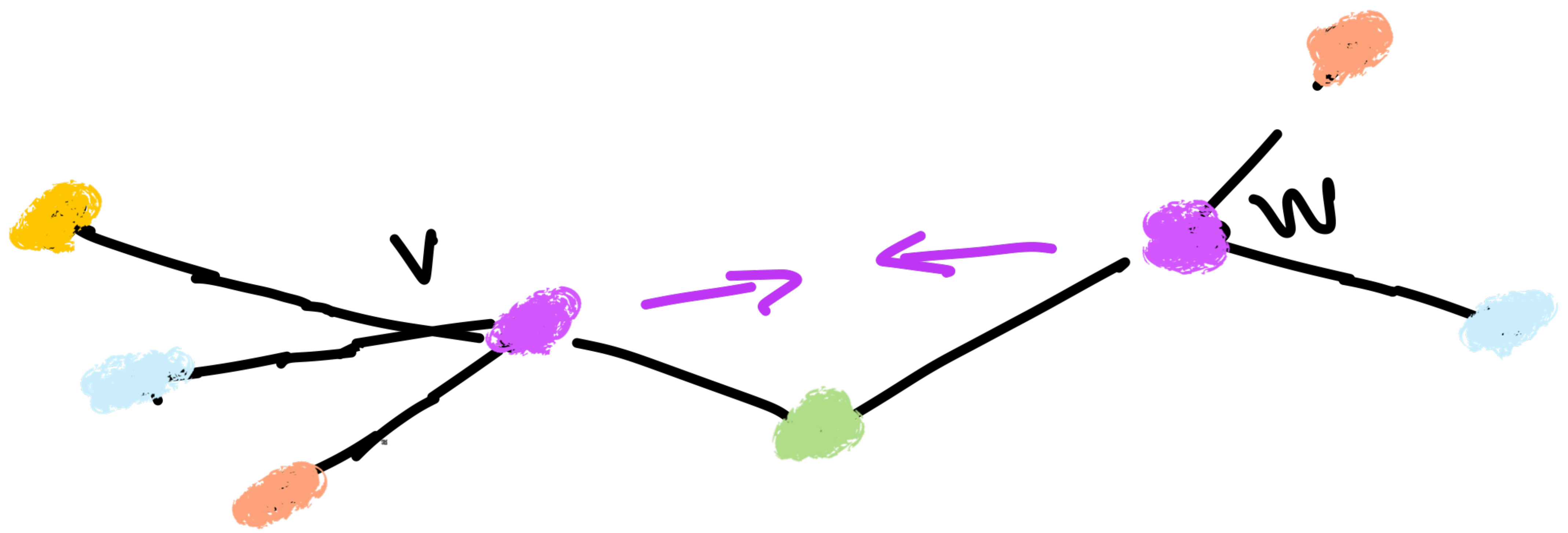

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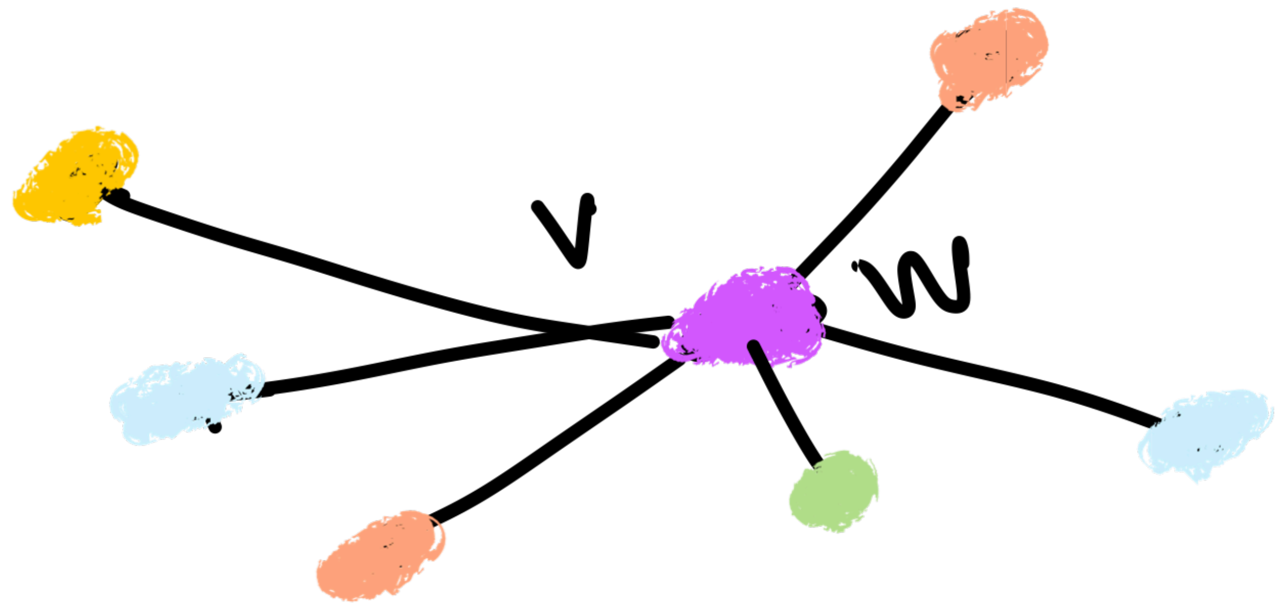
adjacent vertices still have different colors



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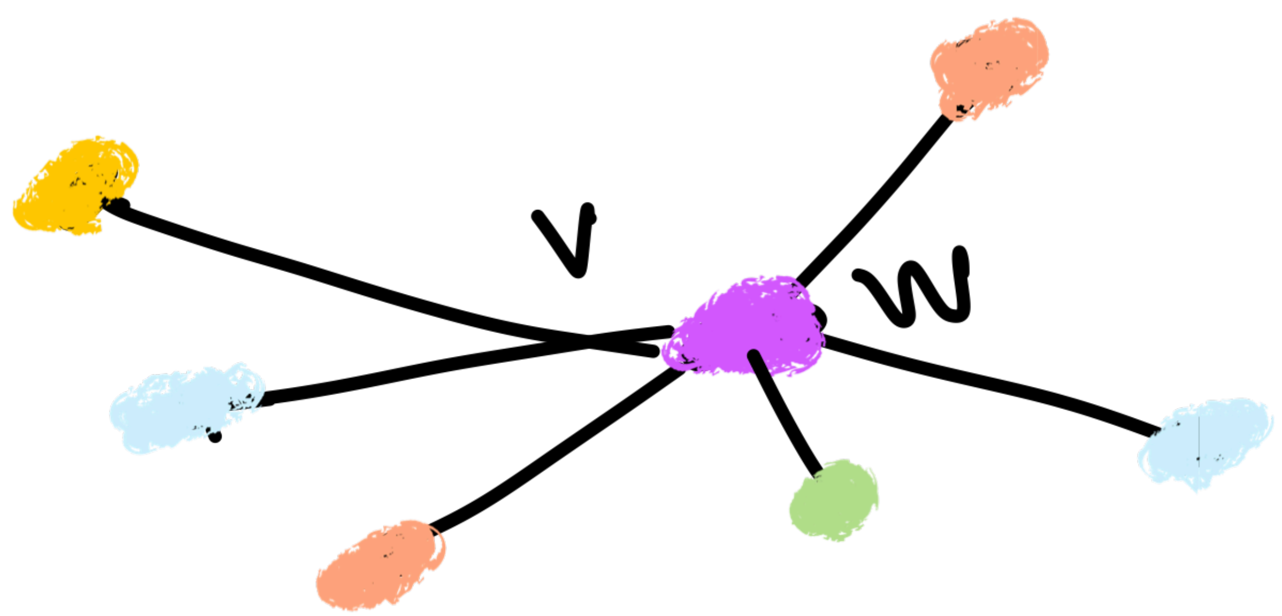

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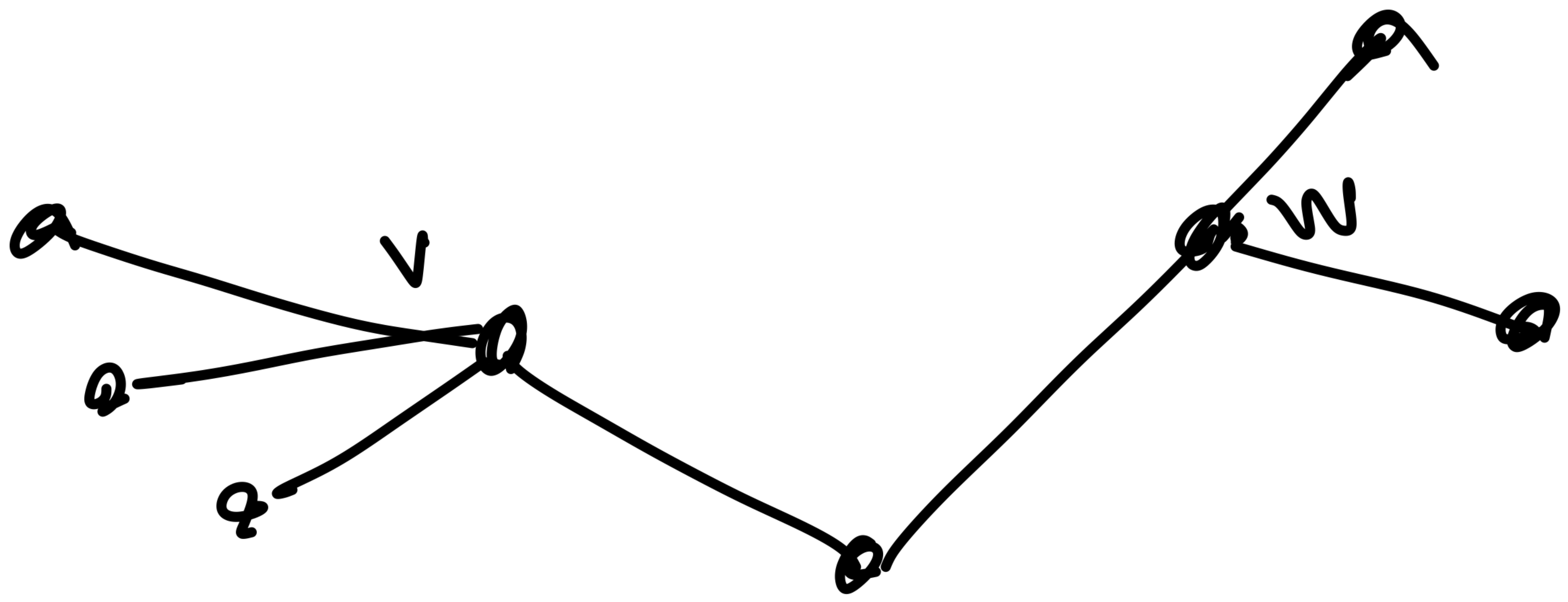
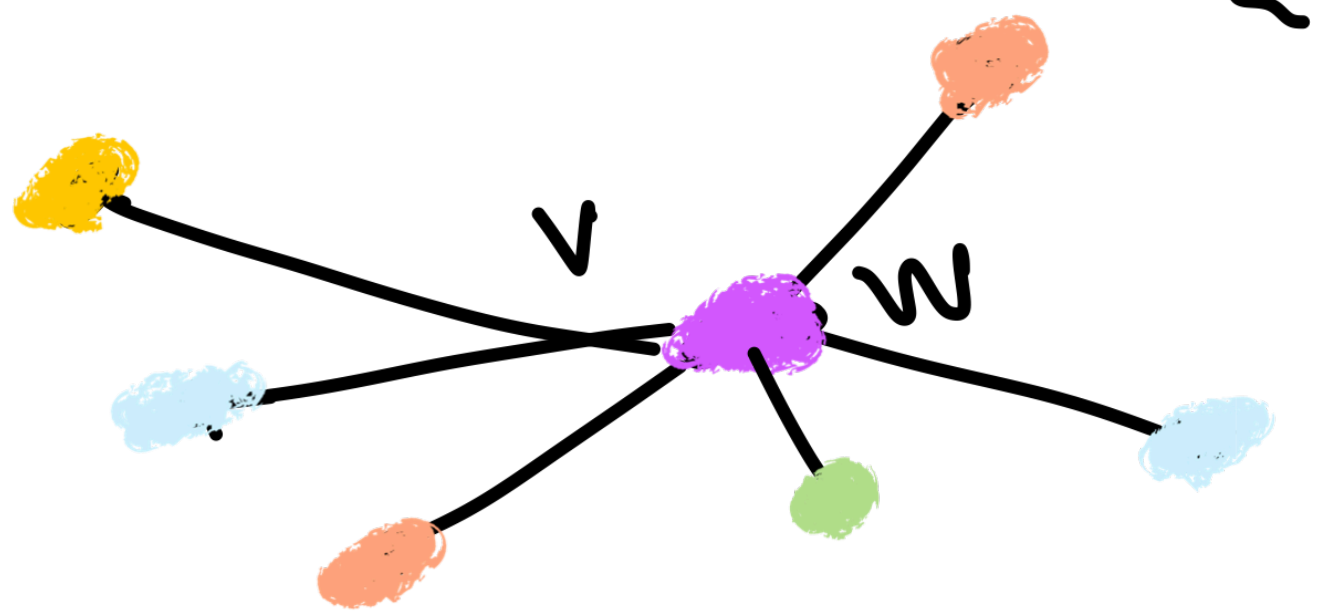


conversely if we have a coloring of G/vw



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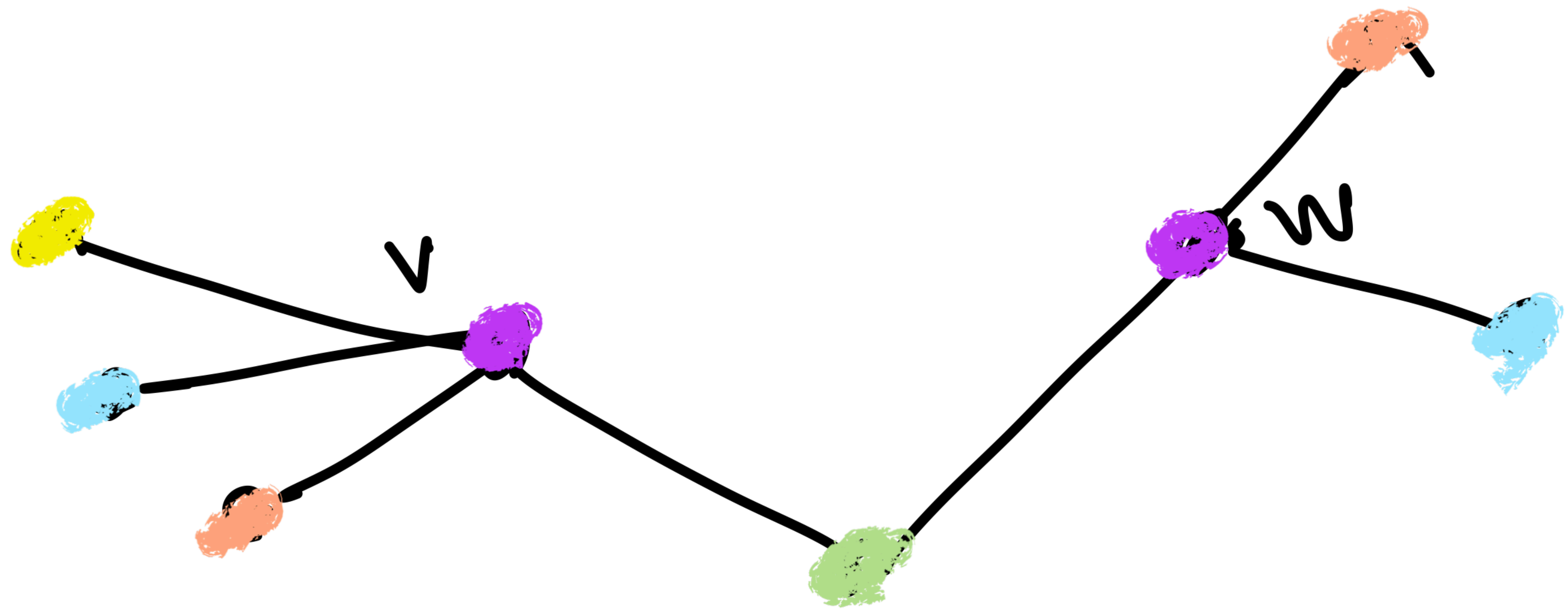
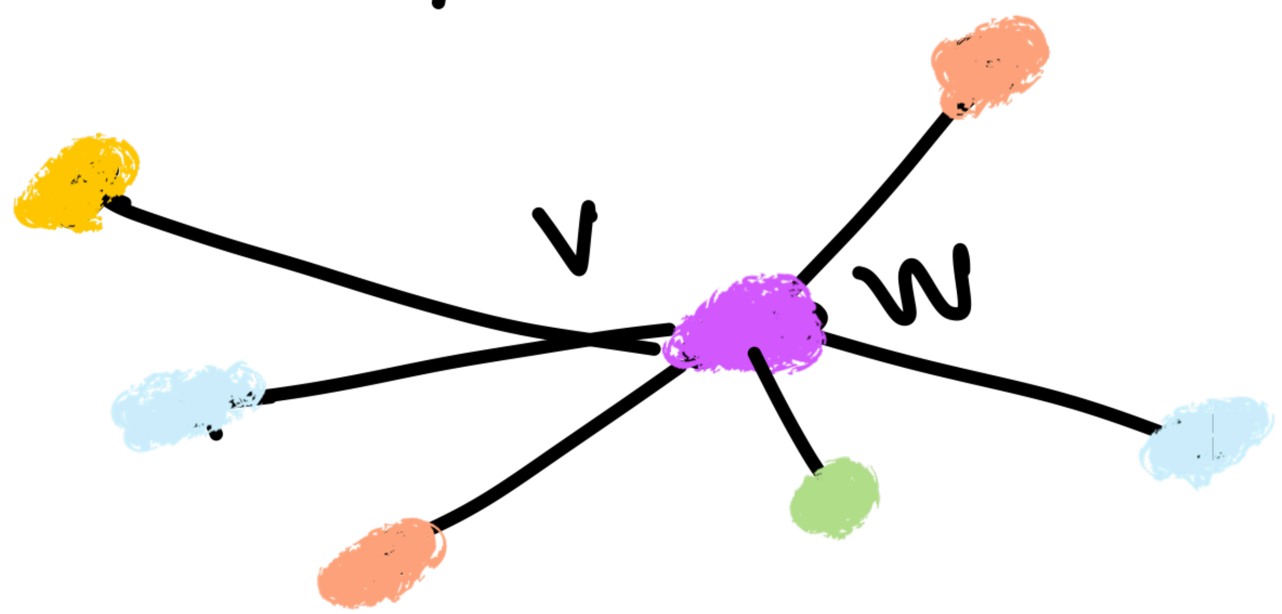
can use colors in original G





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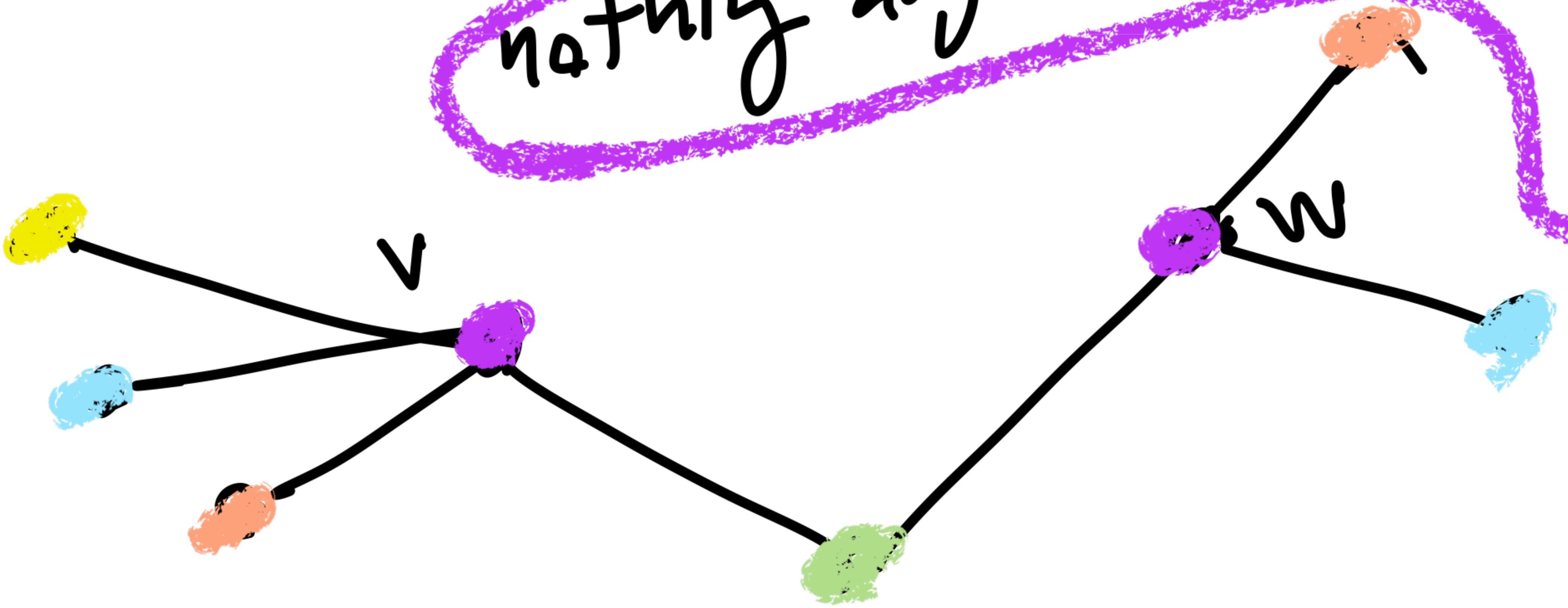
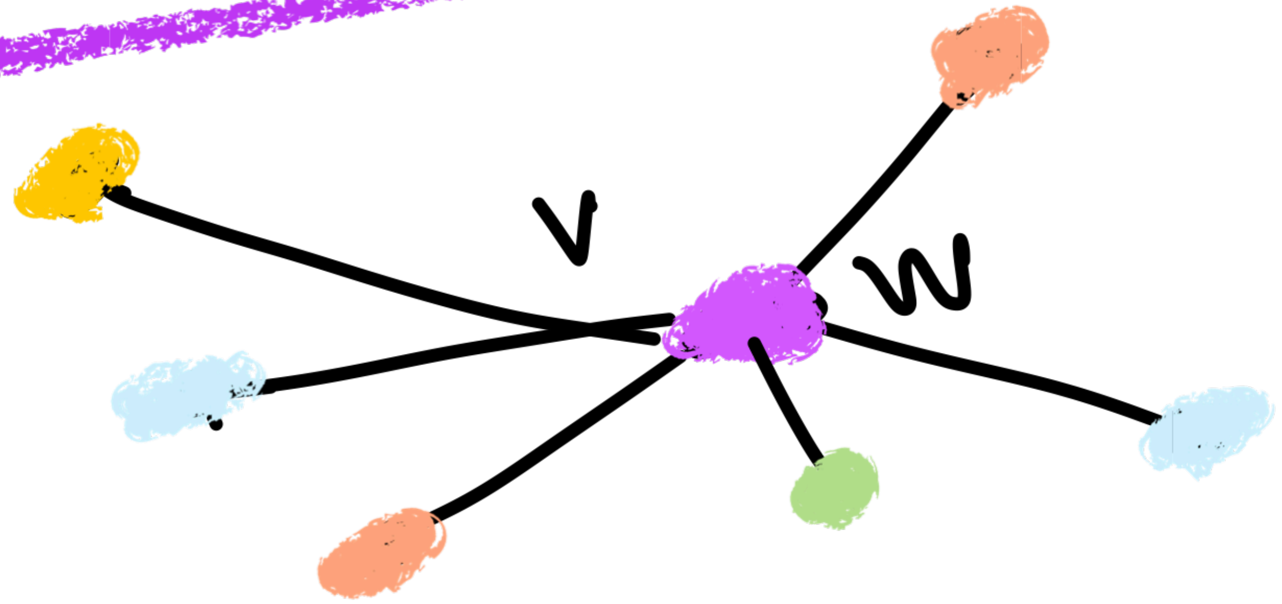
nothing adjacent to vw was purple

⇓

nothing adjacent to v or w is purple!





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$\square$