Towards Brooks' Theorem

Overview and properties of critical graphs

Danny Krashen

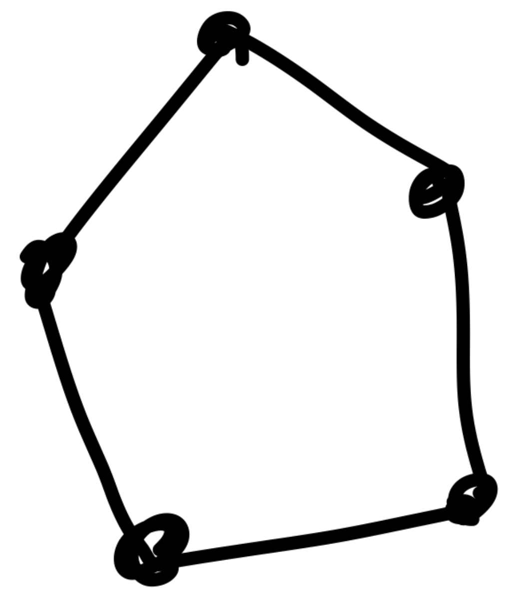
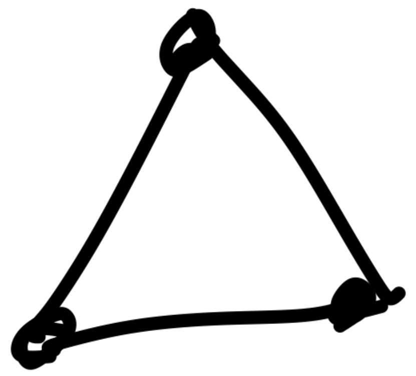
Brooks' Theorem Suppose G is a (simple) connected graph, G not complete and not an odd cycle. Then $\chi(G) \leq \Delta(G)$.

Strategy (Lovász 1973 / Bondy & Murty 1976)

Argue by contradiction, careful examination of a "minimal criminal"

Notion of "minimality"?

Def A graph G is called <u>k-critical</u> if $\chi(G) = k$ but $\chi(G-e) < k$ for all $e \in E(G)$.

ex:  &  are 3-critical

Def G is <u>critical</u> if it is k-critical for some k.

Important properties

1) If $\chi(G) = k$, $\exists H < G$ such that H is k-critical

2) Critical graphs have strong properties – e.g. they are blocks!

3) Can explicitly describe them for $k = 1, 2, 3$.

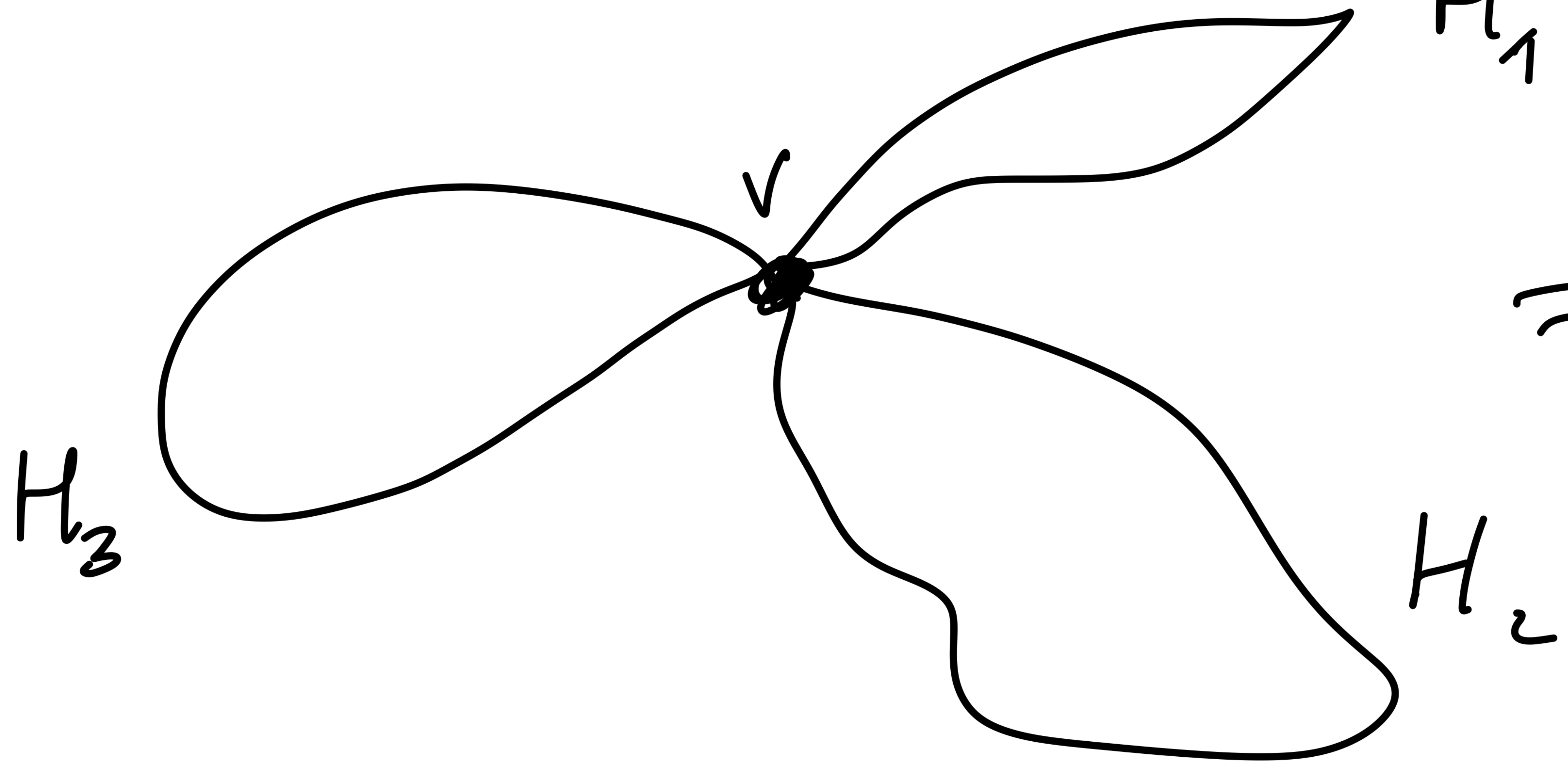
1) If $\chi(G) = k$, why $\exists H < G$ k-critical?

Pf: Consider all subgraphs $H' < G$ such that $\chi(H') = k$. Let H be such a subgraph with fewest possible edges.

Then H is k-critical!

2) Suppose G has a cut vertex.

Then G is not critical.



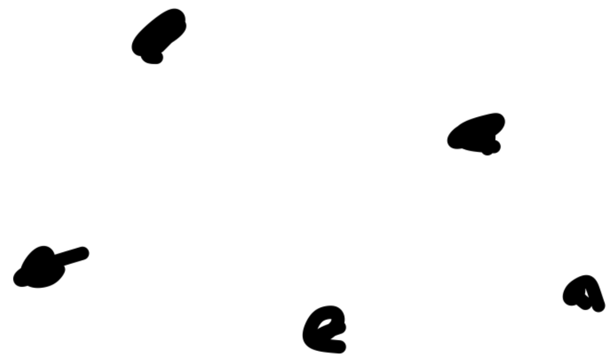
$\Rightarrow \chi(G) = \max_i \{\chi(H_i)\}$

so

$\chi(G) = \chi(H_i)$ some i

3) 1-critical graphs

$\chi(G) = 1 \Rightarrow$ no edges

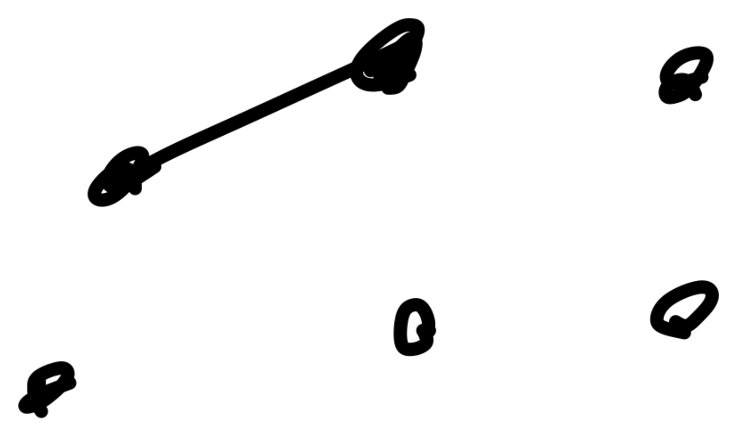
$G =$  trivial

3) 2-critical graphs

$\chi(G) = 2$ but $\chi(G-e) = 1$ all e.

$\Rightarrow$ $\chi(G-e)$ has no edges for <u>any</u> e.

$\Rightarrow$ Can only be one edge. $\chi(G) \neq 1 \Rightarrow$ exactly one edge

$G =$ 

3) 3-critical graphs

$\chi(G)=3$, $\chi(G-e)=2$ all e.

How do we characterize this?

Prop $\chi(H)=2 \iff H$ has no odd cycles.

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Pf: if $C < H$ is an odd cycle $\Rightarrow \chi(C)=3$

$\Rightarrow \chi(H) \geq 3$ ↯

Conversely, if H has no odd cycles, color it:

Start w/ vertex v, give it color

Prop $\chi(H) = 2 \iff H$ has no odd cycles.

Pf: Conversely, if H has no odd cycles, color it:

Start w/ vertex v, give it color

Assume G connected, for any w, color it

if there is a v-w path of even length and

if there is one of odd length.

3) 3-critical graphs

$\chi(G) = 3$, $\chi(G-e) = 2$ all e.

$\implies$ G−e has no odd cycles for every e

& G has an odd cycle.

$\implies$ every edge lies on this odd cycle

$\implies$ G is an odd cycle.