Office: HILL ~~220/207~~

daniel.krashen@gmail.com

Homological Algebra

set of techniques for dealing w/ maps, filtered objects (often graded) in "additive" (= ab. categorical) situations.

replacing objects by their "presentations" described by "complexes"

study of "failure of exactness"

Honest answer:

In 50's topologists developed a set of algebraic techniques for computations.

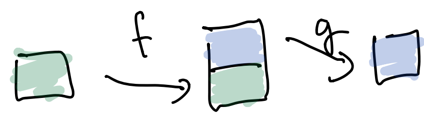
$\rightsquigarrow$ algebra.

Basic object

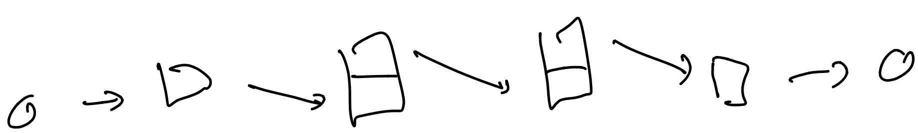
Short exact sequence (Ab groups)

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

- $\ker g = \operatorname{im} f$, f injective, g surjective
- presentation of C
  B generators, A relations
- decomposition of B



Long exact sequences...



Complexes

$\operatorname{im} d \subset \ker d$

$$0 \to A^{-1} \xrightarrow{d^{-1}} A^{0} \xrightarrow{d^{0}} A^{1} \xrightarrow{d^{1}} A^{2} \to A^{3} \to 0$$

"superscripts go up"

$$A^{\bullet} = \coprod A^i \left(= \prod^{\text{or}} A^i \right)$$
$$\overset{\shortparallel}{\bigoplus A^i}$$

$d: A^{\bullet} \circlearrowleft$ shifting degree up by 1.

$$d : A \to A \quad d^2 = 0$$

Where do complexes come from (natural habitats)

presentations / syzygies

$$R = k[x,y] \qquad M = R/(x,y) \cong k$$

$$0 \to (x,y) \to R \twoheadrightarrow k \to 0$$
$$1 \mapsto \bar{1}$$

$$r \mapsto ry, -rx$$
$$a, b \mapsto ax + by$$
$$0 \to R \to R \oplus R \to (x,y) \to 0$$
$$\overset{\shortparallel}{Rx \oplus Ry}$$

"free resolution"

$$0 \to R \to R \oplus R \to R \to M \to 0$$

perspective soon:

$$0 \to 0 \to M \to 0 \to 0$$

equivalence

$$0 \to R \to R \oplus R \to R \to 0$$

<u>example of complexes</u>

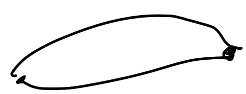
Singular chain complex of a topological space

$$C_2X \to C_1X \to C_0X \to X$$

~~discs (bigons)~~
triangles

equivalences
"
intervals

pts of X



<u>Def</u> A complex $(\to A_i \xrightarrow{d_i} A_{i-1} \to)$

(all complexes potentially infinite w/ 0's extending to right & left)

<u>Def</u> given a complex $C_\bullet$,

$$H_i(C_\bullet) = \frac{\ker d_i}{\operatorname{im} d_{i+1}}$$

Homological notation:
subscripts, $d_i : C_i \to C_{i-1}$

Cohomological notation
superscripts, $d^i : C^i \to C^{i+1}$

$C_\bullet$ chain complex $\quad C^\bullet$ cochain complex

$$H^i(C^\bullet) = \frac{\ker d^i}{\operatorname{im} d^{i-1}}$$
cohomology

<u>Complexes form a category</u>

Start w/ the category ${}_R\mathrm{Mod}$ = left R-modules

if $C_\bullet$ & $D_\bullet$ are complexes of R-modules,

a <u>chain map</u> $f_\bullet : C_\bullet \to D_\bullet$ is a sequence

of maps of R-modules $f_i : C_i \to D_i$

$$f_{i-1}(d_i^C x) = d_i^D (f_i x)$$

or $\quad f_{i-1} d = d f_i$

$$\begin{tikzcd} C_i \arrow[r, "f_i"] \arrow[d, "d_i^C"'] & D_i \arrow[d, "d_i^D"] \\ C_{i-1} \arrow[r, "f_{i-1}"'] & D_{i-1} \end{tikzcd}$$ commutes

Chain map induce natural maps of homology ~~groups~~. R-modules

$$H_i(f_\bullet) : H_i(C_\bullet) \to H_i(D_\bullet)$$

R-module hom.

Can show (try)

Complexes are a category &

$H_i()$ functors cat of Complexes of R-mods to Cat of R-mods

Main magic: given SES of complexes

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

LES $\to H_0(A_\bullet) \to H_0(B_\bullet) \to H_0(C_\bullet) \to H_{-1}(A_\bullet)$

$\to H_{-1}(B_\bullet) \to \cdots$