Injective objects (modules)

"**Recall**" an object I in an Ab cat $\mathcal{A}$ is injective if for any monic $A \hookrightarrow B$ and morphism $A \to I$, $\exists\, B \to I$ s.t. diagram

$$\begin{tikzcd} I & \\ A \arrow[u] \arrow[r, hook] & B \arrow[ul, dashed] \end{tikzcd}$$

commutes.

this is equivalent to $\mathrm{Hom}_{\mathcal{A}}(-, I)$ being exact.

$\mathcal{A}$ has enough injectives if $\forall$ objects $A \in \mathcal{A}$, $\exists$ monic $A \hookrightarrow I$ w/ I injective.

Our goal today: to show that Mod_R has enough injectives.

$\Rightarrow \exists$ univ. δ-functors extending $\mathrm{Hom}_R(-, A)$..

Baer's Criteria

A module M is injective iff $\forall I \trianglelefteq_r R$ (right ideal) and every R-mod map $I \to M$, can extend to $R \to M$.

Pf: Suppose we have $N' \hookrightarrow N$ monic & $N' \to M$, and want to extend to $N \to M$ (& know can do it for $I \to R$). Let $N'' \subset N$ be max'l containing N' s.t. $\exists$ extension $N'' \to M$

$$\begin{tikzcd}
M & & \\
& N'' \arrow[ul, dashed, "f"'] \arrow[dr, hook] & \\
N' \arrow[uu] \arrow[ur, hook] \arrow[rr, hook] & & N
\end{tikzcd}$$

Suppose $N'' \neq N$. Choose $x \in N \setminus N''$

$N''' = N'' + xR$ Let $I = \{r \in R \mid xr \in N''\}$

we have a map $I \to M$ via
$r \longmapsto f(xr)$

by hyp., can extend to $R \xrightarrow{g} M$

now, define $N'' \longrightarrow M$ via

$$n'' + xr \longmapsto f(n'') + g(r)$$

note if $xr \in N''$ then $r \in I \Rightarrow g(r) = f(xr)$

$\square$

Consider case $R = \mathbb{Z}$

an Abelian gp A is injective if every map

$$\begin{tikzcd} A & \\ n\mathbb{Z} \arrow[u] \arrow[r] & \mathbb{Z} \arrow[ul, dashed] \end{tikzcd}$$

extends

i.e.

$$\begin{tikzcd} A & \\ \mathbb{Z} \arrow[u] \arrow[r, "n"] & \mathbb{Z} \arrow[ul, dashed] \end{tikzcd}$$

an elmt $b \in A$ s.t. $nb = a$.

an elmt $a \in A$

i.e. A is injective $\iff$ A is divisible.

Here's one: $\mathbb{Q}/\mathbb{Z}$ (= torsion in S^1)

Exercise:

given an Abelian gp A, consider $\hat{A} = \mathrm{Hom}_{Ab}(A, \mathbb{Q}/\mathbb{Z})$
show: for any $a \in A$ $\exists\, \chi \in \hat{A}$ s.t. $\chi(a) \neq 0$.

Cor:
$$A \hookrightarrow \prod_{\chi \in \hat{A}} \mathbb{Q}/\mathbb{Z}$$
$$a \longmapsto (\chi(a))_{\chi \in \hat{A}}$$

Easy to show: products of injectives are injective.

Recall: If we have cats $\mathcal{A}, \mathcal{B}$ & functors
$\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{B}$ we say L is (left) adjoint to R
or R is (right) adjoint to L if

we have an isom, $\mathrm{Hom}_{\mathcal{A}}(a, Rb) \simeq \mathrm{Hom}_{\mathcal{B}}(La, b)$
natural in both a, b.

$L \vdash R$

Prop if $\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{B}$ w/ $L \dashv R$, and if L is exact, $I \in \mathcal{B}$ is injective, then $R(I)$ is also injective.

Pf: $\mathrm{Hom}_{\mathcal{A}}(-, R(I)) = \mathrm{Hom}_{\mathcal{B}}(L(-), I)$ is a composition of the exact functor L & the exact functor $\mathrm{Hom}_{\mathcal{B}}(-, I)$. $\square$.

Suppose we have a hom. of rings $R \to S$ and let $\mathrm{Mod}_S \xrightarrow{F} \mathrm{Mod}_R$ be the forgetfull functor.

this is exact.

We have adjoints given by:

$$\mathrm{Hom}_R(M, FN) \simeq \mathrm{Hom}_S(M \otimes_R S, N)$$

$$(f : M \to FN) \longmapsto [m \otimes s \mapsto f(m)s]$$

$$\mathrm{Hom}_R(FN, M) \xrightarrow{\sim} \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M))$$
$$(f : FN \to M) \longmapsto [n \mapsto [s \mapsto f(ns)]]$$

Need to show, for every R-module M, can find an injective Ab-gp I and a monomorphism
$$M \to \mathrm{Hom}_{Ab}(R, I)$$

Here's how:

$$\begin{array}{ccc} M \xrightarrow{\sim} \mathrm{Hom}_R(R, M) & \hookrightarrow & \mathrm{Hom}_{\mathbb{Z}}(R, M) \\ & & \downarrow \\ & & \mathrm{Hom}_{\mathbb{Z}}(R, \prod_{x \in \hat{M}} \mathbb{Q}/\mathbb{Z}) \end{array}$$ $\triangleright$.

Now we have:
$$R^i \mathrm{Hom}(-, A) = \mathrm{Ext}^i(-, A)$$

defined via: choose an injective resolution
$$B \hookrightarrow I^0 \hookrightarrow I^1 \hookrightarrow \cdots \qquad I^\bullet = (I^0 \to I^1 \to \cdots)$$

and can apply $\mathrm{Hom}(-,A)$ to I°

$$0 \leftarrow \mathrm{Hom}(I^0,A) \leftarrow \mathrm{Hom}(I^1,A) \leftarrow \cdots$$

$$\underset{\mathrm{Hom}(I^\bullet,A)}{=} \qquad \begin{array}{c} H_i(\mathrm{Hom}(I^\bullet,A)) \\ \shortparallel \\ \mathrm{Ext}^i(B,A) \end{array}$$

<u>Next goal</u>: "balancing"

to show that derived functors of

$\mathrm{Hom}(-,A)$ & $\mathrm{Hom}(B,-)$

and $-\otimes A$ & $B\otimes -$

agree.

in the sense that there is a natural isom

$$R^i\mathrm{Hom}(-,A)(B) \simeq R^i\mathrm{Hom}(B,-)(A)$$