Double complexes

Def: A double complex in $\mathcal{A}$ is a family of objects $\{C_{p,q}\}$ w/ $d_h : C_{p,q} \to C_{p-1,q}$

$$d_v : C_{p,q} \to C_{p,q-1}$$

s.t. $d_h^2 = 0 = d_v^2 = d_v d_h + d_h d_v$

$$\begin{tikzcd}
p,q-2 & & \\
p,q-1 \arrow[u] \arrow[r] & p-1,q-1 & \\
p,q \arrow[u] \arrow[r] & p-1,q \arrow[u] \arrow[r] & p-2,q
\end{tikzcd}$$

$$(d^h + d^v)^2 = 0$$

Note: this is not a complex of complexes (usually)

But if we define $(d^{v'})_{p,q} = (d^v)_{p,q}(-1)^p$

then $d^{v'}$ is a chain map

So $\exists$ correspondence

$$\{\text{dbl complexes}\} \longleftrightarrow \{\text{chain(chain) complexes}\}$$

Def if $\{C_{p,q}\}$ is a double complex, define

$$\mathrm{Tot}^{\prod}(C)_n = \prod_{p+q=n} C_{p,q}$$

$$\mathrm{Tot}^{\amalg}(C)_n = \coprod_{p+q=n} C_{p,q}$$

Note: these are complexes w/ differential $d^h + d^v$

Def $\{C_{p,q}\}$ is **bounded** if only finitely many nonzero terms on each diagonal $p+q=n$.

Goal: $L_n(A \otimes -)(B) = L_n(- \otimes B)(A)$

$$= \mathrm{Tor}_n(A,B)$$

Strategy: Choose proj. resolutions $P_\bullet$ & $Q_\bullet$ for A & B, consider $P_\bullet \otimes Q_\bullet$

$A \otimes Q_\bullet \qquad\qquad P_\bullet \otimes B$

$\mathrm{Tot}(P_\bullet \otimes Q_\bullet)$

$$L_n(-\otimes B)(A) = H_n(P_\bullet \otimes B)$$

$$L_n(A \otimes -)(B) = H_n(A \otimes Q_\bullet)$$

<u>Def</u> Let $\underset{\mathrm{Mod}_R}{\underset{\cap}{P}}$ & $\underset{{}_R\mathrm{Mod}}{\underset{\cap}{Q}}$ be chain complexes.

$P \otimes Q$ double complex w/ $(P\otimes Q)_{p,q} = P_p \otimes Q_q$

$$d^h_{p,q} = d \otimes 1$$

$$d^v_{p,q} = (-1)^p \, 1 \otimes d$$

__Theorem__ (Acyclic Assembly Lemma)

Let $\mathcal{C} = \{C_{p,q}\}$ be a double complex in Mod_R

If

1. C is in upper half plane w/ exact columns

or 2. C is in right half plane w/ exact rows

then $\mathrm{Tot}^{\Pi}(C)$ is acyclic.

If 3. $\mathcal{C}$ is upper half plane w/ exact rows

4. $\mathcal{C}$ right half plane w/ exact columns

then $\mathrm{Tot}^{\amalg}(\mathcal{C})$ is acyclic.

__Pf of (i)__

Assume C is upper half plane, exact columns

want: $\mathrm{Tot}^{\Pi}(\mathcal{C})$ acyclic

Let's show $H_0(\mathrm{Tot}^{\Pi}(\mathcal{C}))$

Suppose $c \in \mathrm{Tot}^{\Pi}(\mathcal{C})_0$ w/ $dc = 0$

$c = (c_{0,0}; c_{-1,1}; c_{-2,2}; \dots)$

want to find $b_{-p,p+1}$ so that

$$d^v(b_{-p,p+1}) + d^h(b_{-p+1,p}) = c_{-p,p}$$

for remainder, read the back

$1 \Rightarrow 4$

upper half exact columns
Π

right half exact columns
$\amalg$

<u>Truncation:</u>

Given a complex $B = \{B_i\}$

define $(\tau_n B)_i = \begin{cases} B_i & \text{if } i > n \\ 0 & \text{if } i < n \\ \ker d_n & \text{if } i = n \end{cases}$

Given $\{C_{pq}\}$ right half plane complex

$1 \Rightarrow 4$ consider $\tau_n^v C$

want to show $\mathrm{Tot}^{\amalg} C$ is acyclic.

know exact columns

Assume $\mathrm{Tot}^{\Pi} D$ is acyclic when D has exact columns & is upper ½

Mapping cones