Index shifting

C complex define $C[p]$:

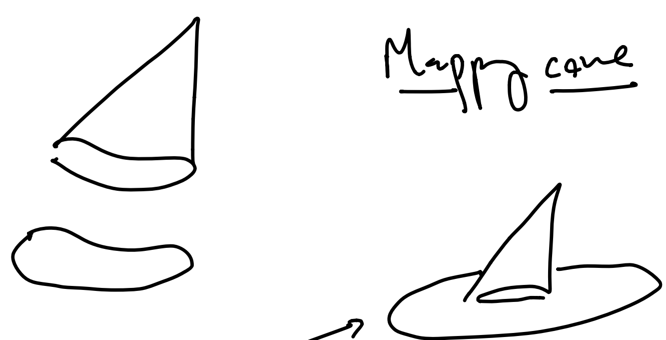
Chain complex $C[p]_n = C_{n+p}$

Cochain complex $C[p]^n = C^{n-p}$

Mapping cones

Cones

Mapping cone



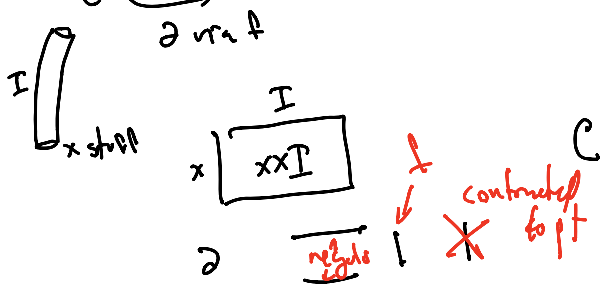
$X \xrightarrow{f} Y$ $\qquad C_f = CX \amalg_f Y$

$X \times I \sqcup Y$

$0 \longrightarrow$ ∂ via f

$CX = X\times I / X \times \{0\}$

$CX \sqcup Y / (x,1) \sim fx$



Def: Let $f: B_\cdot \longrightarrow C_\cdot$ map of chain complexes

cone(f) to be the complex w/

$$\mathrm{cone}(f)_n = B_{n-1} \oplus C_n \quad \text{w/} \quad \partial(b,c) = \left(-\partial(b),\ \partial(c) - f(b)\right)$$

$$\begin{bmatrix} -d_B & 0 \\ -f & d_C \end{bmatrix} : \begin{tikzcd} B_{n-1} \arrow[r, "-d"] \arrow[dr, "-f"] & B_{n-2} \\ \oplus & \oplus \\ C_n \arrow[r, "+d"'] & C_{n-1} \end{tikzcd}$$

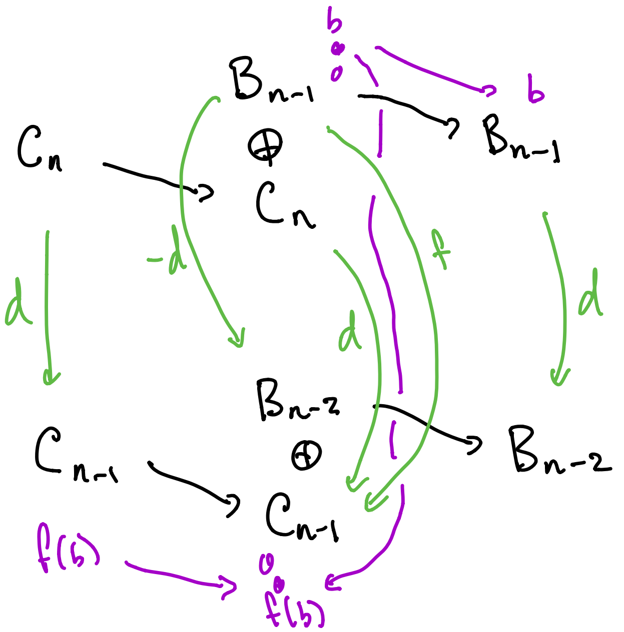
Remark: we have exact seq.

$$0 \to C \longrightarrow \mathrm{cone}(f) \longrightarrow B[-1] \to 0$$

Lem in associated LES for ↑ the boundary map

$$\begin{tikzcd} H_n(C_\cdot) & H_{n+1}(B[-1]) \arrow[l] \\ & H_n(B_\cdot) \arrow[u, phantom, "\shortparallel"] \arrow[ul, bend left, "H_n(f_\cdot)"] \end{tikzcd}$$

Pf:



□.

In particular: f is a q. isom $\iff H_n(\mathrm{cone}(f)) = 0$ all n.

i.e. $\mathrm{cone}(f)$ acyclic.

Back to balancing for & ext

Given $A \in \mathrm{Mod}_R$ $B \in {}_R\mathrm{Mod}$

Choose proj. resolutions P for A & Q for B

$$P_1 \to P_0 \xrightarrow{\varepsilon} A$$

Consider $A \otimes Q$, $P \otimes B$, tot $P \otimes Q$

$$L_n(A \otimes -)(B) = H_n(A \otimes Q) \qquad\qquad L_n(- \otimes B)(A) = H_n(P \otimes B)$$

Goal: show that we have q.isoms

$$A \otimes Q \leftarrow \mathrm{Tot}(P \otimes Q) \rightarrow P \otimes B$$

$$\begin{tikzcd}
A \otimes Q_2 \arrow[d] & P_0 \otimes Q_2 \arrow[l] \arrow[d] & P_1 \otimes Q_2 \arrow[l] \arrow[d] & \arrow[l] \\
A \otimes Q_1 \arrow[d] & P_0 \otimes Q_1 \arrow[l] \arrow[d] & P_1 \otimes Q_1 \arrow[l] \arrow[d] & \arrow[l] \\
A \otimes Q_0 & P_0 \otimes Q_0 \arrow[l] & P_1 \otimes Q_0 \arrow[l] & \arrow[l]
\end{tikzcd}$$

Big double complex (w/ $A \otimes Q_\bullet$) call $C_{\bullet,\bullet}$

$P \otimes Q$

<u>Claim:</u> $\mathrm{Tot}(C)[1] = \mathrm{cone}(\mathrm{Tot}(P \otimes Q) \to A \otimes Q_\bullet)$

$$\begin{tikzcd}[column sep=small]
\mathrm{Tot}(P\otimes Q)_n = \bigoplus_{p+q=n} P_p \otimes Q_q \arrow[d, bend left] & (x,y)_{p,q} \arrow[d] \\
A \otimes Q_n & \underbrace{}_{\substack{0 \text{ if } p \neq 0 \\ \varepsilon(x)\otimes y,\, p=0}}
\end{tikzcd}$$

$\mathrm{Tot}(P \otimes Q) \longrightarrow A \otimes Q_\bullet$

$$\begin{tikzcd}
\mathrm{Tot}(P\otimes Q)_{n-1} \arrow[r] \arrow[dr, "\varepsilon"'] & \mathrm{Tot}(P\otimes Q)_{n-2} \\
\oplus & \oplus \\
A \otimes Q_n \arrow[r] & A \otimes Q_{n-1}
\end{tikzcd}$$

$$\Big(\bigoplus_{p+q=n-1} P_p \otimes Q_q\Big) \oplus (\underset{\substack{\uparrow \\ \mathrm{dg}\,-1}}{A} \otimes Q_n) \qquad D.$$