1. Balancing
2. Group Cohom
3. Sheaf Cohom
4. Spectral Sequences

<u>Last time</u>:

Considered functor in 2 variables

$$\mathrm{Mod}_R \times {}_R\mathrm{Mod} \xrightarrow{\otimes} \mathrm{Ab}$$

given $A \in \mathrm{Mod}_R$ $\quad B \in {}_R\mathrm{Mod}$

consider projective resolutions

$$P_\bullet \to A \qquad Q_\bullet \to B$$

$$P_\bullet \otimes B \longleftarrow \mathrm{Tot}(P_\bullet \otimes Q_\bullet) \longrightarrow A \otimes Q_\bullet$$

q-isoms.

$$\mathrm{Tot}((P_\bullet \to A) \otimes Q_\bullet)$$

In general, suppose that we have a functor of two variables which is left exact in each variable & s.t. when you replace either variable w/ injective object (or proj if contravariant in that variable) the resulting functor of 1 variable is exact.

Such a (bi)functor is called balanced.

<u>Prop</u> (<u>mimicing above proof</u>)

If F is a balanced functor $\mathcal{A} \times \mathcal{A} \to \mathcal{B}$ (or $\mathcal{A}^{op} \times \ldots$), then

$$R^p F(-,B)(A) = R^p F(A,-)(B)$$

"<u>Pf:</u>" choose resolutions

$$A \to I_\bullet \qquad B \to J_\bullet$$

$$F(A,J_\bullet) \leftarrow \mathrm{Tot}\,(F(I_\bullet,J_\bullet)) \to F(I_\bullet,B)$$

modify signs

$\varnothing$.

Application

$$R^p \mathrm{Hom}(A,-)(B) = R^p \mathrm{Hom}(-,B)(A)$$
$$= \mathrm{Ext}^p(A,B)$$

Def Let G be a group, a left G-module is an Abelian group A together w/ a left action of G on A via homomorphisms

i.e. $G \times A \longrightarrow A$ $\quad (g_1 g_2)a = g_1(g_2 a)$

$(g,a) \longmapsto ga$ $\quad 1 \cdot a = a$

$g(a+b) = ga + gb$

Similarly a right G-module is same w/ right action

$A \times G \to A$ $\quad (ag_1)g_2 = a(g_1 g_2)$

$(a,g) \longmapsto ag$

Remark eq. cats G-mods $\longleftrightarrow$ $\mathbb{Z}G$-modules

Def If A is a G-module, we define

invariants: $A^G = \{a \in A \mid ga = a \text{ all } g \in G\}$

coinvariants: $A_G = A / \langle \{ga - a \mid g \in G, a \in A\} \rangle$

Remarks:

$$\begin{tikzcd} G\text{-mod} \arrow[r, bend left, "\text{inv}"] \arrow[r, bend right, "\text{coinv}"'] & Ab \end{tikzcd}$$

l. exact

r. exact

$$\begin{tikzcd} A^G \arrow[r] \arrow[d, hook] & B^G \arrow[d, hook] \\ A \arrow[r] \arrow[d, two heads] & B & G\text{mod} \\ A_G \arrow[r] & B_G \end{tikzcd}$$

$A^G = \mathrm{Hom}_G(\mathbb{Z}, A)$

$(1 \mapsto a) \Rightarrow a \in A^G$

$$\begin{tikzcd} 1 \arrow[r, mapsto] \arrow[d, "g"] & a \arrow[d, "g"] \\ 1 \arrow[r, mapsto] & ga \end{tikzcd}$$

$A_G = A \otimes_{\mathbb{Z}G} \mathbb{Z}$

$\mathbb{Z} \simeq_{\mathbb{Z}G} \mathbb{Z}G / \langle 1 - g \rangle_{g \in G}$

$$\begin{tikzcd} \mathbb{Z}G \arrow[r] \arrow[ur, dashed] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

$g \mapsto 1$

Def $H^n(G,A) = R^n \mathrm{Inv}_G(A)$

$$= R^n \mathrm{Hom}_G(\mathbb{Z}, -)(A)$$

$$= \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$$

Def $H_n(G,A) = L_n \mathrm{CoInv}_G(A)$

$$= L_n(\mathbb{Z} \otimes_{\mathbb{Z}G} -)(A)$$

$$= \mathrm{Tor}^{\mathbb{Z}G}_n(\mathbb{Z}, A)$$

$H^1(G,A) = \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, A)$

Fun Fact $H^n(G,A)$ $\quad$ A trivial action by G

$$\parallel$$

$H^n_{top}(BG, A)$ $\quad$ where BG is a top space s.t.

$\pi_1 BG = G$

and the univ. cov $EG \to BG$ contractible.

ex: $G = \mathbb{Z}$ $\quad$ $BG = S^1$ $\quad$ $\mathbb{R} \to S^1$

EG $\quad$ BG

$$H^n(G,A) = H^n_{top}(S^1, A)$$

Mention: (In standard resolution of $\mathbb{Z}$)

$H^1(G,A)$ $H^2(G,A)$ have well known interpretations

$$H^1(G,A) = \frac{Z^1(G,A)}{B^1(G,A)}$$

$$Z^1(G,A) = \{ \varphi : G \longrightarrow A \mid \varphi(gh) = \varphi(g)\, g(\varphi(h)) \}$$

"crossed homomorphisms"

$B^1(G,A) = \{ \varphi_a : G \longrightarrow A \}$ where

for $a \in A$, $\varphi_a(g) = g(a) - a$

Sheaves

X top space $\mathcal{A}$ Ab cat.

a presheaf on X w/ values in $\mathcal{A}$ is a contravariant functor

$$\mathcal{F} : \mathrm{Op}(X)^{op} \longrightarrow \mathcal{A}$$

$\mathrm{Op}(X)$ = cat. w/ objects open sets in X & morphisms inclusions.

<u>Def</u> A sheaf is a presheaf $\mathcal{F}$ s.t. $\forall$ $\{U_i \to U\}$ open covers in X, we have an equalizer diagram

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \overset{U_i \leftarrow U_i \cap U_j}{\underset{U_j \leftarrow U_i \cap U_j}{\rightrightarrows}} \prod_{(i,j)} \mathcal{F}(U_i \cap U_j)$$

If $U \subset X$ open, we have a functor

$$\begin{array}{rcl} \Gamma(U,-) : \text{Sheaves}/X & \longrightarrow & \mathcal{A} \\ \mathcal{F} & \longmapsto & \mathcal{F}(U) \\ & & \parallel \\ & & \Gamma(U,\mathcal{F}) \end{array}$$

this is left exact, but not always right exact.

<u>Def</u> (when enough injectives)

$$H^n(U,\mathcal{F}) = R^n\Gamma(U,-)(\mathcal{F})$$

<u>Ex</u>: $H^n(X,\mathbb{Z}) \underset{\substack{\text{when} \\ X \text{ is nice}}}{=} H^n_{sing}(X,\mathbb{Z})$

X top space

constant sheaf