Extensions of Groups

Def If A, G groups an extension of A by G is a group E & SES $0 \to A \to E \to G \to 1$ (of groups).

We'll assume A Abelian, use additive notation
mult. notation for G
additive (non Abelian) notation for E.

Observation: in above situation, G acts on A making A a G-module: $\lambda : G \to E$ set-theoretic section

Given $a \in A$, $g \in G$, choose $\lambda g \in E$ a preimage.

$$g \cdot a \equiv \lambda g + a - \lambda g \in \ker(E \to G) = A$$

Check: doesn't depend on λ, since if λ' another lift

$\lambda g - \lambda' g \in A$

$$\lambda g + a - \lambda g$$
$$\shortparallel$$
$$(\lambda g - \lambda' g) + (\lambda' g + a - \lambda' g) - (\lambda g - \lambda' g)$$

$$\lambda g - \lambda' g + \lambda' g + a - \lambda' g + \lambda' g - \lambda g$$
$$\shortparallel$$
$$\lambda g + a - \lambda g \quad \square.$$

From this point of view, can refine question a bit:
Given G & an G-module A, what are possible exts. E, compatible w/ given G-mod structure.

<u>Def</u>: $0 \to A \to E \to G \to 1$ split if can find a section (gp hom) $G \hookrightarrow E$ and in this case, have a semidirect product structure $A \rtimes G \cong E$

To classify E's: strategy is classify "attempted sections"
given any lifting $\lambda: G \to E$ ($\bigstar$ s.t. $\lambda(1) = 0$)

$$\lambda x + \lambda y - \lambda(xy) = [x, y] \in A$$
$$\downarrow \text{into } G$$
$$x\, y\, (xy)^{-1}$$

Defines a function $[\,,\,] : G \times G \longrightarrow A$

"factor set"

<u>Def</u> A factor set from G to A is a map

$G \times G \longrightarrow A$ s.t.

☆ i) $[x,1] = 0 = [1,x]$ (normalized)

2) $x[y,z] - [xy,z] + [x,yz] - [x,y] = 0$

<u>Can check:</u> $[\,,\,]$ defined by λ is a factor set.

$$\lambda x + (\lambda y + \lambda z) = (\lambda x + \lambda y) + \lambda z$$

$$\shortparallel \qquad\qquad\qquad\qquad \shortparallel$$

$$\lambda x + ([y,z] + \lambda(yz)) \qquad \cdots$$

$$\shortparallel$$

$$\lambda x + [y,z] - \lambda x + \lambda x + \lambda(yz)$$

$$x \cdot [y,z] + [x,yz] + \lambda(xyz)$$

Conversely: given any factor set $G \times G \xrightarrow{[\,,\,]} A$

define extension $E_{[\,,\,]}$ via.

$A \times G$

$(a,g)+(a',g') = (a + g\cdot a',\ [g,g'] + gg')$

$a + \lambda g + a' + \lambda g'$

$\shortparallel$

$a + g\cdot a' + \lambda g + \lambda g'$

$= a + g\cdot a' + [g,g'] + \lambda(gg')$

Remark: Factor sets from a group, called

$Z^2(G,A)$

Overcounting: when do two factor sets give same ext?

Given liftings $\lambda, \lambda' : G \to E$

Consider $\underbrace{\lambda' - \lambda}_{\langle ? \rangle} : G \to A \hookrightarrow E$

$[\,,\,]_\lambda \quad [\,,\,]_{\lambda'}$

Claim is: $[g,h]_{\lambda'} - [g,h]_{\lambda} = g\langle h\rangle - \langle gh\rangle + \langle g\rangle$

$$\lambda' g + \lambda' h = [g,h]_{\lambda'} + \lambda'(gh) = [g,h]_{\lambda'} + \langle gh\rangle + \lambda(gh)$$

$$= \langle g\rangle + \lambda g + \langle h\rangle + \lambda h$$

$$= \langle g\rangle + g\cdot\langle h\rangle + \lambda g + \lambda h$$

$$= \langle g\rangle + g\cdot\langle h\rangle + [g,h]_{\lambda} + \lambda(gh)$$

$$[g,h]_{\lambda'} + \langle gh\rangle = \langle g\rangle + g\langle h\rangle + [g,h]_{\lambda}$$

$$[g,h]_{\lambda'} - [g,h]_{\lambda} = g\langle h\rangle - \langle gh\rangle + \langle g\rangle$$

Prop $[\,,\,], [\,,\,]' \in Z^2(G,A)$ give same ext

iff $\exists\ \langle\,\rangle : G \to A$ s.t. arbitrary function.

$$[g,h]' - [g,h] = g\langle h\rangle - \langle gh\rangle + \langle g\rangle$$

all g,h

Def: if $\langle\ \rangle : G \to A$ arbitrary,

set $\partial\langle\ ,\ \rangle : G\times G \to A$ defined by

$$\partial\langle g,h\rangle = g\langle h\rangle - \langle gh\rangle + \langle g\rangle$$

Claim: $\partial\langle\ ,\ \rangle \in Z^2(G,A)$

Pf: consider $E = A \rtimes G$ choose $\lambda : G \to E$

$\lambda(g) = (\langle g\rangle, g)$

$[\ ,\ ]_\lambda = \partial\langle\ ,\ \rangle$

Def $B^2(G,A) = \{\partial\langle\ ,\ \rangle \mid \langle\ \rangle : G \to A\}$

Theorem there is a bijection

$$\{\text{exts } 0 \to A \to E \to G \to 1\}/\text{isom} \longleftrightarrow \frac{Z^2(G,A)}{B^2(G,A)} = H^2(G,A)$$

Rem: $(0 \to A \to E \to G \to 1) \xrightarrow{\sim} (0 \to A \to E' \to G \to 1)$

is a isom $E \xrightarrow{\sim} E'$ s.t. diag. commutes.

Suppose $e: 0 \to A \to E \to G \to 1$ is ~~an~~ a split ext.
and $\varphi \in \mathrm{Aut}(e)$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "1"] & E \arrow[r] \arrow[d, "g \mapsto ag"] & G \arrow[r] \arrow[d, "1"] & 1 \\
0 \arrow[r] & A \arrow[r] & E \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

$$g \longmapsto \langle g\rangle + g \qquad \langle\ \rangle : G \to A$$

$$gh \longmapsto \langle gh\rangle + gh$$
$$= \langle g\rangle + g + \langle h\rangle + h$$
$$= \langle g\rangle + g\cdot\langle h\rangle + \underbrace{g+h}_{gh}$$ split.

$$\langle gh\rangle = \langle g\rangle + g\langle h\rangle$$

__Def__: Crossed hom $G \xrightarrow{\langle\,\rangle} A$ is a function s.t.

Set of crossed homs is gp under $+$ in A

notation $Z^1(G,A)$

One ex of aut of E is an inner one.

Turns out that inner Auts$(E) \cap$ Aut(e)

corresp to $\langle\,\rangle : G \to A$

$\partial a : g \longmapsto ga - a$ some ~~fixed~~ $a \in A$