Last time:

"examples" from gp cohomology: extensions

Main idea to connect yesterday w/ gp cohom is the "Bar resolution" of $\mathbb{Z}$ as a G-module.

explicitly will write a canonical res

$$\cdots \to B_1 \to B_0 \to \mathbb{Z}$$

Def: B_i = free $\mathbb{Z}G$-mod w/ basis G^i

write $(g_1, \ldots, g_i)$ as $[g_1 | \cdots | g_i]$

B_0 = free $\mathbb{Z}G$-mod w/ basis $[\ ]$

$$d_1[x] = x[\ ] - [\ ] = (x-1)[\ ]$$

$$d_2[x|y] = x[y] - [xy] + [x]$$

$$d_3[x|y|z] = x[y|z] - [xy|z] + [x|yz] - [x|y]$$

$$d_i[x_1|\cdots|x_i] = x_1[x_2|\cdots|x_i] + \sum_{j=1}^{i-1} (-1)^j [x_1|\cdots|x_j x_{j+1}|\cdots|x_i] + (-1)^i [x_1|\cdots|x_{i-1}]$$

$$Z^1(G,A) = \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z},A)$$

$$B_2 \to B_1 \to B_0 \to \mathbb{Z}$$

$$\mathrm{Hom}(B_2, A) \leftarrow \mathrm{Hom}(B_1,A) \leftarrow \mathrm{Hom}(B_0,A)$$

B_1 = free gen $[x]$

$G \xrightarrow{f} A$ set map

$[x] \longrightarrow f(x)$

$$B_2 \longrightarrow B_1 \xrightarrow{[f]} A$$

$[x|y] \longrightarrow x[y] - [xy] + [x]$

$\downarrow [f]$

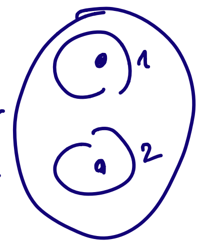
$x f(y) - f(xy) + f(x)$

$f(xy) = f(x) + x f(y)$

mult notation

$f(gh) = f(g)\, g(f(h))$

Some comments about Sheaf cohomology

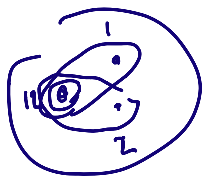
X 

sheaf $\mathcal{F}$ on X

given by $\mathcal{F}_1 = \mathcal{F}(\odot)$

$\mathcal{F}_2 = \mathcal{F}(\odot)$

$$\mathcal{F}(X) = \mathcal{F}_1 \times \mathcal{F}_2$$



$$\begin{tikzcd}
 & \mathcal{F}_1 \arrow[dr] & \\
\Gamma = \mathcal{F}(X) \arrow[ur] \arrow[dr] & & \mathcal{F}_{12} \\
 & \mathcal{F}_2 \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & Ab \\
Sh(X) \arrow[ur, "stalk(x)"] \arrow[r] & Ab
\end{tikzcd}$$

$$\mathcal{F} \longrightarrow \mathcal{F}(X) = \Gamma(X, \mathcal{F})$$

Spectral Sequences

Given a complex (C_n, d)

Goal: compute $H_n(C_\bullet)$

Notational shortcut: $C = \bigoplus_{n \in \mathbb{Z}} C_n$

$$d : C \to C$$

$$H(C) = \frac{\ker d}{\operatorname{im} d}$$

Assume that we're given a filtration of C respecting d's

i.e. subobjects $F_n C \subset F_{n-1} C$

$$F_{n+1} C \subset F_n(C) \ldots$$

$$d F_i C \subset F_i C$$

For simplicity, say

$$0 = F_N C \subset F_{N-1} C \subset \cdots \subset F_0 C = C$$

$$F_i C / F_{i+1} C$$

$$\mathcal{A} \xrightarrow{G} \mathcal{B} \xrightarrow{F} \mathcal{C} \qquad FG$$

$A \to \mathcal{A}^\circ$ $\qquad$ $G \mathcal{A}^\circ$

$$\begin{tikzcd}
 & & \\
FJ_{0,0} \arrow[r] \arrow[u] & FJ_{1,0} \arrow[r] \arrow[u] & FJ_{2,0} \arrow[u] \\
GA \arrow[r] & FGI_0 \arrow[r] \arrow[u] & FGI_1 \arrow[r] \arrow[u] & FGI_2 \arrow[ul]
\end{tikzcd}$$

Given our filtration of C,

consider $F_i C / F_{i+1} C = gr_i C$ $\quad d$ induces a d_i on this

can compute $H(gr_i C)$

alternatively, $Z(C) = \ker d$

$$F_i Z(C) = F_i C \cap Z(C)$$

$$F_i H(C) = \text{image of } F_i Z(C) \text{ in } H(C)$$

get $gr_i H(C)$

question

how to compare $gr_i H(C)$ vs $H(gr_i C)$

Want $\qquad$ Have

$$
\begin{tikzcd}
F_i \mathcal{C} \arrow[r, "d"] & F_i \mathcal{C} \\
& F_{i+1}\mathcal{C} \arrow[u, phantom, "\cup" description] \\
& \vdots \arrow[u, phantom, "\cup" description]
\end{tikzcd}
$$

$$Z_{i,j}\mathcal{C} = \{ a \in F_i\mathcal{C} \mid da \in F_{i+j}\mathcal{C} \}$$

$$F_i = Z_{i,0}\mathcal{C} \supset Z_{i,1}\mathcal{C} \supset \dots \supset Z_{i,N}\mathcal{C} = Z(\mathcal{C}) \cap F_i\mathcal{C}$$

$$
\begin{tikzcd}
F_i\mathcal{C} \arrow[r, "F_i d"] & \overset{Z_{i,0}}{F_i\mathcal{C}} \arrow[r, "F_i d"] & F_i\mathcal{C} \\
F_{i+1}\mathcal{C} \arrow[r] \arrow[u, phantom, "\cup" description] & F_{i+1}\mathcal{C} \arrow[r] \arrow[u, phantom, "\cup" description] & F_{i+1}\mathcal{C} \arrow[u, phantom, "\cup" description]
\end{tikzcd}
$$

$$\frac{\overset{Z_{i,1}}{(F_i d)^{-1}(F_{i+1}\mathcal{C})}}{\mathrm{im}(F_i d)} \twoheadrightarrow H(\mathrm{gr}_i \mathcal{C})$$

$F_i = F_i C$

$D_i = H(F_i) \qquad E_i := H(F_i/F_{i+1})$

$0 \to F_{i+1} \to F_i \to F_i/F_{i+1} \to 0$

$$\begin{tikzcd} H(F_{i+1}) \arrow[r] & H(F_i) \arrow[r] & H(F_i/F_{i+1}) \arrow[ll, bend left] \end{tikzcd}$$

$$\begin{tikzcd} D_{i+1} \arrow[rr] & & D_i \arrow[dl] \\ & E_i \arrow[ul] & \end{tikzcd}$$

$$\begin{tikzcd} E_i \arrow[r] \arrow[rr, bend right] & D_{i+1} \arrow[r] & E_{i+1} \end{tikzcd}$$

$H(gr_i C) \to H(gr_{i+1} C)$

this map "checks" if $d\alpha \in F_{i+2}$

get a sequence of maps

$$H(\underbrace{H(gr_i C)}_{H(gr_i C) \to H(gr_{i+1} C)})$$

in limit

$H(E) \ldots (H(g_i^n(C)))$

approaches $g_i H(C)$