Spectral Sequences (Attempt 2)

Topological spaces

X top space $Z \subset X$ subspace

$H_n(X)$ $\qquad\qquad \dfrac{Z_n(X)}{B_n(X)}$

$Z_n^1(X) =$ elmts α of $C_n(X)$ s.t. $d\alpha \in C_{n-1}(Z)$

similarly computing part of $H_n(X)$ whose classes can be represented as cycles supported in Z (i.e. im. of $H_n(Z)$ in $H_n(X)$)

$(C_{n+1}X) \cap d^{-1}(C_n Z) \to$

$C_{n+1}Z \to Z_n(Z) \to H_n(Z)$

1st approximation of $H_n X$ "+ quot. is next"

$\cup$

part from $H_n Z$

approx pnt for im $H_n Z = H_n Z$

appr. pnt for quot. is $\dfrac{Z_n^?(X)}{B_n(X)}$

$H_n^?(X) \subset H_n(X)$ $\quad 0 \to H_n^?(X) \to H_n(X) \to H_n(X)/H_n^?(X) \to 0$

"

cycles supp in Z

$$0 \to C(Z) \to C(X) \to C(X)/C(Z) \to 0$$

$$C(X)/C(Z) \cong C(X,Z)$$

LES:

$$H_{n+1}(X,Z) \to H_n(Z) \to H_n(X) \to H_n(X,Z) \to H_{n-1}(Z)$$

in general, consider a filtration of subs

$$\emptyset = X_0 \subset X_1 \subset \dots \subset X_n = X$$

$F_i H_n X$ = part supported in X_i

i.e. image of $H_n X_i$

$$F_i H_n X \big/ F_{i-1} H_n X$$

What's a spectral sequence?

<u>Def</u> A unindexed spectral sequence in $\mathcal{A}$ is a sequence of objects & endomorphisms

$(E_r, d_r)_{r \geq a}$ $d_r : E_r \to E_r$

$$d_r^2 = 0$$

together w/ isoms. $H(E_r, d_r) \cong E_{r+1}$

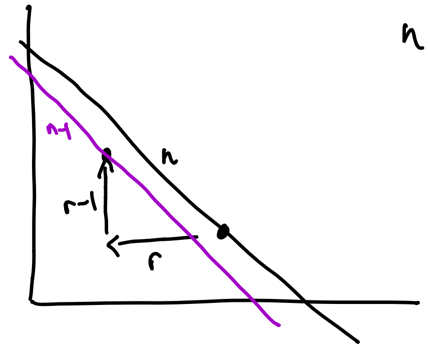
with two indices

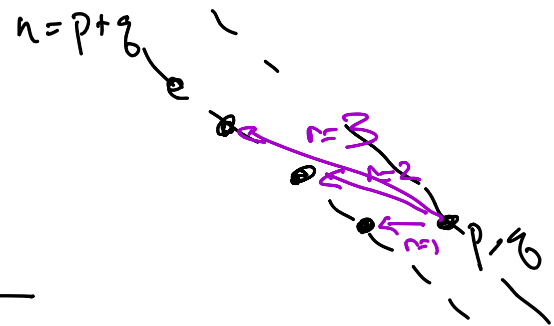
<u>Def</u> A spectral sequence of homological type is a family of objects $\{E^r_{p,q}\}_{r \geq a}$

and morphisms $d^r_{p,q} : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$

s.t. $d_r^2 = 0$ & have isoms

$$E^{r+1}_{p,q} = \frac{\ker d^r_{p,q}}{\operatorname{im} d^r_{p+r,q-r+1}}$$





Def A spectral sequence of cohomological type is a family of objects $\{E_r^{p,q}\}_{r\geq a}$ and morphisms $d_r^{p,q}: E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$ s.t. $d_r^2=0$ & have isoms

$$E_{r+1}^{p,q} = \frac{\ker d_r^{p,q}}{\operatorname{im} d_r^{p-r,q+r-1}}$$

$$E_r^{p-r,q+r-1} \to B_r^{p,q} \subset E_r^{p,q} \supset Z_r^{p,q} = \ker(d_r \longrightarrow\ )$$

$B_r^{p,q}$ $\qquad\qquad$ $Z_r^{p,q}$

$$Z_r^{p,q}/B_r^{p,q} = E_{r+1}^{p,q}$$

by corresp. thm, have $B_{r,r+1}^{p,q} \subset Z_{r,r+1}^{p,q}$

in $E_r^{p,q}$

$B_r^{p,q} \subset B_{r,r+1}^{p,q} \subset \cdot \qquad \subset Z_{r,r+1}^{p,q} \subset Z_r^{p,q}$

sim. corr. to $E_{r+2}^{p,q}$

(in $E_r^{p,q}$)

$B_r^{p,q} \subset B_{r,r+1}^{p,q} \subset B_{r,r+2}^{p,q} \subset \cdots \qquad \subset Z_{r,r+2}^{p,q} \subset Z_{r,r+1}^{p,q} \subset Z_r^{p,q}$

$B_\infty^{p,q} = \varinjlim_r B_{a,r}^{p,q} \qquad Z_\infty^{p,q} = \varprojlim Z_{a,r}^{p,q}$

$E_\infty^{p,q} = Z_\infty^{p,q} / B_\infty^{p,q}$

In general, given A_0 $\quad A_1$ subquot of A_0

$A_2 \quad \cdot \quad - \quad A_1$

$\vdots$

$$A_i = B_{i-1}/C_{i-1}$$

$$C_{i-1} \subset B_{i-1} \subset A_{i-1}$$

$$A_2 = B_1/C_1 \qquad C_1 \subset B_1 \subset A_1$$

$$C_1 \subset B_1 \subset A_1 = B_0/C_0 \subset A_0/C_0$$

$$C_1 = \tilde{C}_1/C_0 \qquad B_1 = \tilde{B}_1/C_0$$

$$C_0 \subset \tilde{C}_1 \subset \tilde{B}_1 \subset B_0$$

Def We say $E^r_{p,q}$ is bounded if for all n only finitely many terms $E^r_{p,q}$ $p+q=n$ are nonzero.

in this case, $B^{a,r}_{p,q}$ stabilize after finite number of steps.

& $Z^{a,r}_{p,q}$

Definition We say that a spectral seq $\{E^r_{p,q}, d^r\}$ converges to $\{H_n\}_{n\in\mathbb{Z}}$ (w/r/to a filtration $\subseteq F_p H_n \subseteq F_{p+1} H_n \cdots$)

notation $E^a_{p,q} \Rightarrow H_{p+q}$ if

we are given isoms $E^\infty_{p,q} \simeq F_p H_{p+q} / F_{p-1} H_{p+q}$

often will see $E^a_{p,q} \Rightarrow H_n$

Similarly

$$E_a^{p,q} \Rightarrow H^{p+q} \quad \text{means have}$$

a filtration

$$\subseteq F^{p+1}H \subseteq F^p H^n \subseteq F^{p-1} H \subseteq$$

s.t. $\quad E_\infty^{p,q} \simeq F^p H^{p+q} / F^{p+1} H^{p+q}$

$$\begin{tikzcd}
0 & 0 & & & & \\
0 & 0 & E^2_{0,1} \arrow[ull] & E^2_{1,1} \arrow[ull] & E^2_{2,1} & \\
& & E^2_{0,0} \arrow[ull] & E^2_{1,0} \arrow[ull] & E^2_{2,0} \arrow[ull] & E^2_{3,0} \arrow[ull] \\
& & & & 0 \arrow[ull] & 0 \arrow[ull]
\end{tikzcd}$$

$$E^2_{p,q} \Rightarrow H_{p+q}$$

$$E^2_{1,0} = E^\infty_{1,0}$$

$$E^2_{2,0} \to E^2_{0,1} \to E^3_{0,1} = E^\infty_{0,1} \to 0$$

$$0 \to E^\infty_{0,1} \to H_1 \to \underset{\substack{\parallel \\ E^2_{1,0}}}{E^\infty_{1,0}} \to 0$$

$$0 \to E^{\infty}_{2,0} \to E^{2}_{2,0} \to E^{2}_{0,1} \to H_1 \to E^{2}_{1,0} \to 0$$

$$0 \to E^{\infty}_{1,1} \to H_2 \to E^{\infty}_{2,0} \to 0$$

$$0 \to E^{\infty}_{1,1} \to H_2 \to E^{2}_{2,0} \to E^{2}_{0,1} \to H_1 \to E^{2}_{1,0} \to 0$$