Hyperderived functors

Given $F : \mathcal{A} \longrightarrow \mathcal{B}$ additive, right exact

$\mathcal{A}$ has enough projectives

constructed $L_i F : \mathcal{A} \longrightarrow \mathcal{B}$

$$L_i F(A) = H_i(F(P_\bullet))$$

$P_\bullet \to A$ projective resolution.

Today, will define

$$\mathbb{L}_i F : Ch(\mathcal{A}) \longrightarrow \mathcal{B}$$

(or bounded complexes...)

SES of chain complexes $\longrightarrow$ LES in $\mathcal{B}$, etc.

$$\mathbb{L}_i F(A) = L_i F(A)$$

A = complex in deg 0

<u>Def</u> $\mathcal{A}$ has enough projectives, $A_\cdot \in Ch(\mathcal{A})$.

A left Cartan-Eilenberg resolution for $A_\cdot$ is a double complex $P_{\cdot\cdot}$ of projectives w/ $P_{pq}=0$ $q<0$ & $P_{p0} \xrightarrow{\varepsilon} A_p$ "augmentation" s.t. vertical objects form proj. resl'ns of A_p's.

$$\begin{array}{cc} P_{0,1} & \vdots \\ \downarrow & \\ P_{0,0} & \downarrow \\ \downarrow & \downarrow \\ A_0 & A_1 \end{array}$$

s.t.

1) $P_{p,*}=0$ if $A_p=0$

2) horiz. boundaries, cycles & hom gps are projective res'lns of boundaries, cycles, hom gps for A's.

$$P_{p-1,q} \xleftarrow{d^h} P_{p,q} \xleftarrow{d^h} P_{p+1,q}$$

$$B^h_{p,q} = \operatorname{im}(P_{p+1,q} \xrightarrow{d^h} P_{p,q})$$

$$Z^h_{p,q} = \ker(\qquad)$$

$$H^h_{p,q} = Z/B$$

we require

$$\begin{array}{ccc}
B^h_{p,q} & Z & H \\
\downarrow d^v & \downarrow & \downarrow \\
B^h_{p,q-1} & \vdots & \vdots \\
\downarrow & & \\
\vdots & & \\
\downarrow d^h & & \\
B_{p,0} & & \\
\downarrow & & \\
B_p & &
\end{array}$$

are all proj. resolutions.

$\leftarrow$ fr $A_\bullet$

<u>Lemma</u> These exist.

Proof:

Consider SES

$$0 \to B_p \to Z_p \to H_p \to 0$$

(in $A_\bullet$)

choose proj resolutions of B_p & H_p

$$P^B_{p*} \quad P^H_{p*}$$

$$\begin{tikzcd} P^B_{p*} \arrow[r] \arrow[d] & P^Z_{p*} \arrow[r] \arrow[d] & P^H_{p*} \arrow[d] & \\ 0 \arrow[r] & B_p \arrow[r] & Z_p \arrow[r] & H_p \arrow[r] & 0 \end{tikzcd}$$

Horseshoe lemma.

but now, consider another horseshoe:

$$\begin{tikzcd} P^Z_{p*} \arrow[r] \arrow[d] & P_{p*} \arrow[r, "\text{"}d\text{"}"] \arrow[d] & P^B_{p-1,*} \arrow[d] & \\ 0 \arrow[r] & Z_p \arrow[r] & A_p \arrow[r, "d"] & B_{p-1} \arrow[r] & 0 \end{tikzcd}$$

horizontal differentials given by

$$\begin{tikzcd} P_{pq} \arrow[r] \arrow[d] \arrow[rrr, bend left, "d^h"] & P^B_{p-1,q} \arrow[r] \arrow[d] & P^Z_{p-1,q} \arrow[r] \arrow[d] & P_{p-1,q} \arrow[d] \\ P_{p,q-1} \arrow[r] & P^B_{p-1,q-1} \arrow[r] & P^Z_{p-1,q-1} \arrow[r] & P_{p-1,q-1} \end{tikzcd}$$

commutes, get a complex of complexes

after sign trick $\rightsquigarrow$ double complex.

Exercise (5.7.2)

Show that if $A_\cdot \xrightarrow{f} B_\cdot$ is a chain map & $P_{\cdot\cdot} \to A_\cdot$, $Q_{\cdot\cdot} \to B_\cdot$ CE resolutions of A & B then $\exists$ a map of double complexes $\tilde{f} : P \to Q$ which extends to a map of "augmented" double complexes

<u>Def</u> Suppose $f, g : D_{\cdot\cdot} \to E_{\cdot\cdot}$ are maps of double complexes. a chain homotopy $s : f \to g$ is a collection of maps

$$s^h_{p,q} : D_{p,q} \to E_{p+1,q}$$

$$s^v_{p,q} : D_{p,q} \to E_{p,q+1}$$

$$g - f = (d^h s^h + s^h d^h) + (d^v s^v + s^v d^v)$$

$$\text{and} \quad 0 = s^v d^h + d^h s^v = s^h d^v + d^v s^h$$

<u>Exercise:</u> Chain homotopies $s : f \to g$, $f, g : D_{\cdot\cdot} \to E_{\cdot\cdot}$ induce chain homotopies

$$\mathrm{Tot}^{\oplus}(s) : \mathrm{Tot}^{\oplus}(f) \to \mathrm{Tot}^{\oplus}(g)$$

$\mathrm{Tot}^{\oplus}(f), \mathrm{Tot}^{\oplus}(g) : \mathrm{Tot}^{\oplus}(D) \to \mathrm{Tot}^{\oplus}(E)$

<u>Exercise (5.7.3)</u>

1) If $f, g : A_\bullet \to B_\bullet$ maps of chain complexes,

$P_{\bullet\bullet} \to A_\bullet$, $Q_{\bullet\bullet} \to B_\bullet$ CE resolutions,

$\tilde{f}, \tilde{g} : P_{\bullet\bullet} \to Q_{\bullet\bullet}$ "extensions" of f & g

then $\tilde{f}$ chain homotopic to $\tilde{g}$

2) Show that any two CE resolutions of a chain complex $A_\bullet$ are chain hom. equivalent.

<u>Cor</u>: if P, Q are CE resolutions of A, and F additive, $\mathrm{Tot}^{\oplus}(FP) \sim_{hom} \mathrm{Tot}^{\oplus}(FQ)$

<u>Def</u> Let $F : \mathcal{A} \to \mathcal{B}$ be right exact, $\mathcal{A}$ has enough projectives. Given $A \in \mathrm{Ch}(\mathcal{A})$, choose $P_{\bullet\bullet} \to A_\bullet$ CE resolution. We define $\mathbb{L}_i F(A)$

$= H_i(\mathrm{Tot}^{\oplus} F(P))$

Facts (exercises?)

Let $Ch_{\geq 0}(\mathcal{A})$ be chain complexes $\{A_\cdot\}$

s.t. $A_p = 0$ for $p < 0$.

$F : \mathcal{A} \to \mathcal{B}$

Then $\mathbb{L}_0 F(A) = H_0(F(A))$

and $\mathbb{L}_i F(P_\cdot) = 0 \quad i > 0$

split - projective ~~ed~~ ?

$\Rightarrow \mathbb{L}_i F$ unin. δ functor

$Ch_{\geq 0}(\mathcal{A}) \to \mathcal{B}$

$\Rightarrow$ they are left derived functors of

$A \mapsto H_0(F(A))$

$\Rightarrow$ SES of bounded complexes $\to$ LES in $\mathcal{B}$