Fix a ring R (associative, unital, not necessarily commutative)

Let ${}_R\mathrm{Mod}$ cat. of left R-modules
Mod_R " right "

Remark ${}_R\mathrm{Mod}$ is equivalent to the category $\mathrm{Mod}_{R^{op}}$

unfair question: is ${}_R\mathrm{Mod}$ isomorphic to $\mathrm{Mod}_{R^{op}}$?

Typical convention: Use Mod_R

$f: M \to N$ is a right R-module map
if $f(mr) = (fm)r$

Def We say that a sequence of maps

$A \xrightarrow{f} B \xrightarrow{g} C$ is exact (at B) if
$\operatorname{im} f = \ker g$.

Def exact sequence of multiple maps
$A_{i+1} \to A_i \to A_{i-1} \cdots$ means exact at each A_i.

Def SES$_R$ is the category whose objects are short exact sequences and whose morphisms are comm. diagrams

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

Things in common w/ Mod$_R$

Def Ab-Category is a category $\mathcal{A}$ together with the structure of an Abelian group on every Hom set. $\mathrm{Hom}_{\mathcal{A}}(A,B)$ is an Ab gp

s.t. $f\circ(g+g') = fg + fg'$
$(g+g')h = gh + g'h$.

Observation SES$_R$ is an Ab-Cat, (Mod$_R$ is also) (induced by adding maps in each component)

Def if $\mathcal{A}, \mathcal{B}$ are Ab-Cats, A functor $F: \mathcal{A} \to \mathcal{B}$ is additive if $\forall A, B \in ob(\mathcal{A})$, the map

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(A,B) &\longrightarrow \mathrm{Hom}_{\mathcal{B}}(FA,FB) \\ (f : A \to B) &\longmapsto (Ff : FA \to FB) \end{aligned}$$ is an Ab. gp homomorphism.

<u>Exercise</u> ☆ Suppose $\mathcal{A}$ is an Ab-Cat, $\mathcal{C}$ is a Cat.

Consider the category $\mathrm{Fun}(\mathcal{C},\mathcal{A}) = \mathcal{F}$

$\mathrm{Hom}_{\mathcal{F}}(F,G) = \{$ natural transformations $\alpha : F \to G\}$

Given $\alpha, \beta : F \to G$ define $\alpha+\beta : F \to G$ via

$A \in \mathcal{C}$ $\alpha : F \to G$

$\alpha(A) : F(A) \to G(A)$

$A \xrightarrow{f} B$

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha(A)} & G(A) \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\alpha(B)]{} & G(B) \end{array}$$

$(\alpha+\beta)(A) = \alpha(A)+\beta(A)$

Show that this gives $\mathcal{F} = \mathrm{Fun}(\mathcal{C},\mathcal{A})$ the structure of an Ab-Cat.

<u>Examples</u> $\mathcal{C} = \bullet$ $\mathcal{A} = \mathrm{Mod}_R$

$\mathcal{C} = \bullet \to \bullet$ $A \xrightarrow{f} B$

<u>Ex</u>? ☆ same w/ $\mathcal{C}$ Ab-Cat $\mathcal{F} = \mathrm{AddFun}(\mathcal{C},\mathcal{A})$

Def An additive Category is an Ab-Cat $\mathcal{A}$ s.t.
- $\exists$ a 0-object in $\mathcal{A}$ (initial & final)
- $A \times B$ exists for any $A, B \in ob(\mathcal{A})$

Examples Mod_R, SES_R, $Fun(\mathcal{C}, \mathcal{A})$, $Fun_{Add}(\mathcal{C}, \mathcal{A})$

Ab-Cat

$\mathcal{A}$ an add cat

Def A chain complex in $\mathcal{A}$, $\mathcal{A}$ is an Ab-Cat is a collection of objects $\{A_i\}_{i \in \mathbb{Z}} = A_\bullet$ & morphisms $d_i : A_i \to A_{i-1}$ s.t.

$$d_{i-1} d_i = 0 \in Hom_{\mathcal{A}}(A_i, A_{i-2})$$

Def A morphism of chain complexes $A_\bullet \xrightarrow{f_\bullet} B_\bullet$ in $\mathcal{A}$ is a sequence of morphisms $A_i \xrightarrow{f_i} B_i$ such that $\forall i$, the diagram

$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & B_i \\ d_i \downarrow & & \downarrow d_i \\ A_{i-1} & \xrightarrow[f_{i-1}]{} & B_{i-1} \end{array}$$

commutes.

Note: if $\mathcal{A}$ is an Ab-Cat then so is $Ch(\mathcal{A})$
(component-wise)

if $\mathcal{A}$ is an Additive Cat so is $Ch(\mathcal{A})$

$$Ch(\mathcal{A}) = Fun_{Add}\left(K, \mathcal{A}\right)$$

$$K = \left[\begin{tikzcd} \ar[r] & K_i \ar[r, "d"] \ar[rr, bend left, "0"] \ar[loop below, "\mathbb{Z}"] & K_{i-1} \ar[r, "d"] \ar[rr, bend right, "0"'] \ar[loop below, "\mathbb{Z}"] & K_{i-2} \ar[r] \ar[loop below, "\mathbb{Z}"] & {} \end{tikzcd}\right]$$

Def Let A be an additive category, $f: B \to C$

Then

- The kernel of f is a morphism $K \to B$ such that
$$\begin{tikzcd} K \ar[r] \ar[rr, bend right, "0"'] & B \ar[r, "f"] & C \end{tikzcd}$$
and such that

K is universal with this property in the sense that if $K' \to B$ is any morphism s.t.
$$\begin{tikzcd} K' \ar[r] \ar[rr, bend right, "0"'] & B \ar[r, "f"] & C \end{tikzcd}$$
then $\exists$ unique $K' \to K$

such that the diagram

$$\begin{tikzcd} K' \arrow[r] \arrow[rr, bend right] & K \arrow[r] & B \end{tikzcd}$$ commutes.

we write $K = \ker f$

Alternately, $K = \varprojlim \left(\begin{tikzcd} B \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & C \end{tikzcd} \right)$

- The cokernel of f is a morphism $C \to D$ s.t. s.t. the composition $\begin{tikzcd} B \arrow[r, "f"] \arrow[rr, bend right, "0"'] & C \arrow[r] & D \end{tikzcd}$ is 0 and which is universal for this in the sense that if $C \to D'$ w/ $\begin{tikzcd} B \arrow[r] \arrow[rr, bend right, "0"'] & C \arrow[r] & D' \end{tikzcd}$ then $\exists !$ morphism $D \to D'$ s.t.

$$\begin{tikzcd} C \arrow[r] \arrow[rr, bend right] & D \arrow[r] & D' \end{tikzcd}$$ commutes.

Alternately $D = \operatorname{coker}(f) = \varinjlim \left(\begin{tikzcd} B \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & C \end{tikzcd} \right)$

- We say that f is <u>monic</u> if for every $B' \xrightarrow{g} B$ w/ $\begin{tikzcd} B' \arrow[r, "g"] \arrow[rr, bend right, "0"'] & B \arrow[r] & C \end{tikzcd}$ we have $g = 0$. ($\ker = 0$)

- We say that f is <u>epic</u> if for any $C \xrightarrow{g} C'$ w/ $B \to C \xrightarrow{g} C'$ we have $g = 0$
$$\begin{tikzcd} B \arrow[r] \arrow[rr, bend right, "0"'] & C \arrow[r, "g"] & C' \end{tikzcd}$$
(coker $= 0$)

<u>Exercise</u> ☆ Show that SES_R need not have kernels or cokernels.

- LES_R (RES_R) be cat. of left (right) exact short sequences

sequences
$0 \to A \to B \to C$
$A \to B \to C \to 0$

<u>Q</u>: Do either of these have kernels or cokernels?

<u>Def</u> An Abelian Category is an additive category $\mathcal{A}$ such that:
- every morphism has a kernel & a cokernel
- every monic is the kernel of its cokernel
- every epic is the cokernel of its kernel

$$A \overset{\text{monic}}{\hookrightarrow} B \qquad B \to B/A$$

$$\ker(B \to B/A) = A$$

$B \to B/A$ epic

kernel $A \to B$

<u>Prop.</u> if $\mathcal{A}$ is an Ab. category so is $\mathrm{Fun}_*(\mathcal{C}, \mathcal{A})$ and so is $\mathrm{Ch}(\mathcal{A})$

<u>Exercise*</u> if we consider SES_R a subcat of $\mathrm{Ch}(\mathrm{Mod}_R)$ then the smallest Abelian subcat of $\mathrm{Ch}(\mathrm{Mod}_R)$ containing SES_R & containing all objects of $\mathrm{Ch}(-)$ iso. to its objects is $\mathrm{Ch}(\ )$.