Recall from last time:

Hyperderived functors

$F : \mathcal{A} \to \mathcal{B}$ right exact functor

define $\mathbb{L}_i F : Ch(\mathcal{A}) \to \mathcal{B}$

Idea of construction:

Given $A_\bullet \in Ch(\mathcal{A})$, find a new complex $P_\bullet$ of projective objects together with a q. isom $P_\bullet \to A_\bullet$, then define $\mathbb{L}_i F(A_\bullet) = H_i(F P_\bullet)$

(compare: if $A \in \mathcal{A}$, $P_\bullet \to A$ a res of proj)

then

$$\begin{array}{ccccccc} \to & P_1 & \to & P_0 & \to & 0 & \to 0 \\ & & & \downarrow & & \downarrow & \\ 0 & \to & & A & \to & 0 & \to 0 \end{array}$$

More specifically, we constructed $P_\bullet = \mathrm{Tot}^{\oplus} P_{\bullet\bullet}$ where $P_{\bullet\bullet}$ is an Eilenberg-Maclane resolution of A.

$\mathbb{L}_i F(A) = H_i(\mathrm{Tot}^{\oplus} F P_{\bullet\bullet})$

Recall: given any double complex $C_{\bullet\bullet}$ then can consider filtrations of $\mathrm{Tot}^{\oplus} C_{\bullet\bullet}$ (via horiz or vertical)

Filtrations:

$$({}^{I}\tau_{\leq n} C_{\bullet\bullet})_{p,q} = \begin{cases} C_{p,q} & \text{if } p \leq n \\ 0 & \text{else} \end{cases}$$

$$({}^{II}\tau_{\leq n} C)_{p,q} = \begin{cases} C_{p,q} & \text{if } q \leq n \\ 0 & \text{else} \end{cases}$$

$${}^{I}E^{0}_{p,q} = C_{p,q} \qquad {}^{I}E^{1}_{p,q} = H^{v}_{q}(C_{p,*})$$

$${}^{I}E^{2}_{p,q} = H^{h}_{p} H^{v}_{q}(C)$$

(assumed for convergence that C was bounded)

Notational clash:
we would like to use p for two different things
- filtration index for spectral seq
- horiz coord (of double complex).

Conclusion (following convention)
filtration wins: when discussing ss for a 2nd filtration on dbl complex, we write coords as $C_{q,p}$.

$$^{II}F_p \operatorname{Tot}^{\oplus} C = \bigoplus_{j \le p} C_{i,j}$$

$$^{II}F_p \operatorname{Tot} C \Big/ {}^{II}F_{p-1} \operatorname{Tot} C = \bigoplus_i C_{ip} = \text{"} C_{*p} \text{"}$$

$$^{II}E^0_{p,q} = C_{qp} \qquad {}^{II}E^1_{p,q} = H^h_q(C_{*,p})$$

$$^{II}E^2_{p,q} = H^v_p H^h_q(C)$$

Left hyperderived functor,

$$\mathbb{L}_i F(A) = H_i(\mathrm{Tot}^{\oplus} F P_{\cdot\cdot})$$

$${}^{I}E^1_{p,q} = H^v_q(FP_{p,*})$$

$$\begin{array}{ccc}
 & FP_{p,q+1} & \\
 & \downarrow & \\
\leftarrow & FP_{p,q} & \leftarrow \\
 & \vdots & \\
 & \downarrow & \\
\leftarrow & FP_{p,0} & \leftarrow \\
 & \downarrow & \\
\leftarrow & FA_p & \leftarrow
\end{array}$$

$L_q F(A_p)$

get maps

$$L_q F(A_{p-1}) \leftarrow L_q F(A_p) \leftarrow$$

from applying $L_q F$ to $A_\bullet$

$${}^{I}E^2_{p,q} = H^h_p H^v_q (FP_{\cdot\cdot}) = H^h_p L_q F(A_*)$$
$$= H_p(L_q F(A_\bullet))$$

would want to say

$${}^{I}E^2_{p,q} = H_p(L_q F(A_\bullet)) \Rightarrow \mathbb{L}_{p+q} F(A_\cdot)$$

want dbl cplx bounded

implied by $A_\bullet$ bounded from below
$A_\bullet \in Ch_{\geq 1}(\mathcal{A})$

Go back in time to the
Classical Convergence Theorem (Thm 5.5.1)
Suppose $C_\bullet$ is a complex, $F_\bullet$ is a filtration on $C_\bullet$
recall: have a sseq $E^1_{p,q} = H_{p+q}(gr^F_p C)$
Q: when does this converge to $H_{p+q}(C)$?

1. If F is bounded then s.seq. is bounded & have convergence

[recall F bounded means (all n,) $F_i C_n = F_{i-1} C_n$ if $i \gg 0$ or $i \ll 0$
$E^n_{p,q}$ bounded means $\forall n$, $E^n_{p,q} = 0$ if $p+q=n$, $|p|, |q| \gg 0$]

2. If F is bounded from below and exhaustive then sseq is bounded from below & have convergence.

F b. from below means $F_{i-1}C_n = F_i C_n$ $i \ll 0$

E b. . . . . . all n $E^r_{p,q} = 0$ if $p \ll 0$
$p+q=n$

F exhaustive means

$$\varinjlim_p F_p C_n \cong C_n$$

Oops — from now on, filtration is $F_\bullet$, functor is $\mathcal{F}$

Note: ${}^{I}F_\bullet \operatorname{Tot} \mathcal{F}P_{\bullet\bullet}$ is bounded if $A_\bullet \in Ch_{\geq}(\mathcal{A})$
but if $A_\bullet$ is unbounded, ${}^{I}F$ need not even be bounded from below.

But ${}^{II}F_\bullet \operatorname{Tot} \mathcal{F}P$ is bounded from below, so gives a convergent sseq always.

Conclusion

$$(L_p{}^{\circ}\mathcal{F})(H_q A_\bullet) = {}^{II}E^2_{p,q} \Rightarrow \mathbb{L}_{p+q}\mathcal{F}(A_\bullet)$$

$${}^{II}E^{0}_{p,q} = \mathcal{F} P_{q,p} \qquad {}^{II}E^{1}_{p,q} = H^{h}_{q}(\mathcal{F} P_{*,p})$$

$$\underset{?}{=} L_p \mathcal{F} A_q$$